A Canonical Model of the Region Connection Calculus

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Abstract

Canonical models are very useful for determining simple representation formalism for qualitative relations. Allen's interval relations, e.g., can thereby be represented using the start and the end point of the intervals. Such a simple representation was not possible for regions of higher dimension as used by the Region Connection Calculus. In this paper we present a canonical model which allows regions and relations between them to be represented as points of the topological space and information about their neighbourhoods. With this formalism we are able to prove that whenever a set of RCC-8 formulas is consistent there exists a realization in any dimension, even when the regions are constrained to be (sets of) polytopes. For three- and higher dimensional space this is also true for internally connected regions. Using the canonical model we give algorithms for generating consistent scenarios.

1 Introduction

The Region Connection Calculus (RCC) is a topological approach to qualitative spatial representation and reasoning [5] where spatial regions are subsets of topological space [18]. Of particular interest for application purposes is RCC-8, a sub-calculus of RCC that uses eight mutually exhaustive and pairwise disjoint

base relations. The computational properties of RCC-8 have been studied thoroughly [17, 19] and efficient reasoning mechanisms were identified.

Despite this, there are still some problems with representational aspects of RCC. As the calculus is based on topology, an obvious canonical model is the topological space. However, it appears to be difficult to represent such models in a computational framework. Allen's interval relations [1], for example, where the real numbers are a canonical model, can be represented as a pair of two real numbers, the start and the end point of an interval of real numbers. Such a simple representation was not possible for regions of higher dimensions as used by RCC.

Another representational drawback of using RCC is that topological space does not have a particular dimension whereas most applications deal only with two- or three-dimensional space. It might be possible that a set of RCC formulas is consistent but not realizable within a particular dimension. Lemon [16] gave an example of a set of spatial formulas which is realizable in three dimensional space but not in two dimensional space, if regions are constrained to be internally connected. Lemon uses this result to argue that spatial logics like RCC are not an adequate formalism for representing space.

In this paper we will refer to these representational topics and present a canonical model of RCC-8 that permits a simple representation of regions by reducing regions to their necessary topological features. Based on this we will prove that for any consistent set of spatial formulas there are in fact realizations in any dimension $d \geq 1$ when regions are allowed to be internally disconnected. This is still true even when regions are constrained to be sets of polytopes. Actually, internal connectedness of regions is not at all forced in the RCC-theory, so RCC can still be seen as an adequate representation formalism of space. We will also argue that forcing internal connectedness of all regions is too restrictive when dealing with spatial regions. Nevertheless, we will prove that in three- and higher dimensional space regions can be allowed to be internally connected.

The remainder of the paper is structured as follows. In Section 2 we will introduce RCC-8 and some basic topological terms. Section 3 sketches the modal encoding of RCC-8 and presents the canonical model of RCC-8. In Section 4 we give a topological interpretation of this model which is used in Section 5 to prove that there is always a realization in any dimension. Section 6 describes how models of a set of spatial relations can be determined and how a consistent scenario can be generated. In Section 7 we will discuss our results.

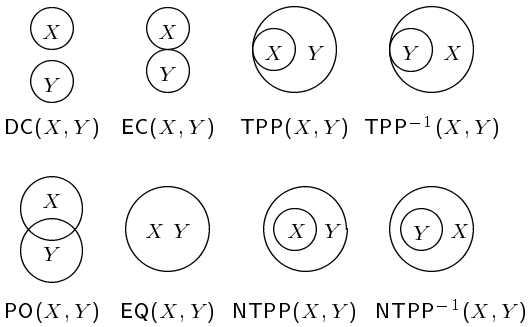

Figure 1: Two-dimensional examples for the eight base relations of RCC-8.

2 Qualitative Spatial Representation with RCC

RCC is a topological approach to qualitative spatial representation and reasoning where *spatial regions* are subsets of topological space [18]. Relationships between spatial regions are defined in terms of the relation $C(A, B)$ which is true iff the closure of region A is connected to the closure of region B, i.e., if they share a common point. Regions themselves do not have to be internally connected, i.e., a region may consist of different disconnected parts. The domain of *spatial variables* (denoted as X, Y, Z) is the whole topological space.

RCC-8 uses a set of eight pairwise disjoint and mutually exhaustive relations, called *base relations*, denoted as DC, EC, PO, EQ, TPP, NTPP, TPP^{-1}, and NTPP^{-1}, with the meaning of *DisConnected, Externally Connected, Partial Overlap, EQual, Tangential Proper Part, Non-Tangential Proper Part*, and their converses. Examples for these relations are shown in Figure 1.

Sometimes it is not known which of the eight base relations holds between two regions, but it is possible to exclude some of them. In order to represent this, unions of base relations can be used. Since base relations are pairwise disjoint, this results in 2^8 different relations. A *spatial formula* is a relation between two particular regions, a *spatial configuration* is a set Θ of spatial formulas. Θ is *consistent*, if it is possible to find a *realization* of Θ, i.e., a *consistent scenario* where every spatial variable is instantiated by a spatial region such that all relations hold between the regions.

As we will go further into topology, we will define some relevant terms:

Definition 2.1 *Let $\mathcal{U}$ be a set, the universe. A* topology *on $\mathcal{U}$ is a family T of subsets of $\mathcal{U}$, with*

1. *if O_1, $O_2 \in T$, then $O_1 \cap O_2 \in T$,*

2. *if* $O_i \in T$ *for* $i \in I$, *then* $\bigcup O_i \in T$,

3. $\emptyset, \mathcal{U} \in T$.

A topological space *is a pair* $\langle \mathcal{U}, T \rangle$. *Every subset* $O \subset \mathcal{U}$ *with* $O \in T$ *is* open.

If the particular topology T on a set is not important then the topological space $\langle \mathcal{U}, T \rangle$ will be denoted simply by $\mathcal{U}$.

Definition 2.2 *Let* $\mathcal{U}$ *be a topological space,* $M \subset \mathcal{U}$ *be a subset of* $\mathcal{U}$ *and* $p \in \mathcal{U}$ *be a point in* $\mathcal{U}$.

- M *is* closed *if* $\mathcal{U} \setminus M$ *is open.*
- *A set* $N \subset \mathcal{U}$ *is said to be a* neighbourhood *of* p *iff there is an open subset* $O \subset \mathcal{U}$ *such that* $p \in O \subset N$.
- p *is said to be an* interior point *of* M *if there is a neighbourhood* N *of* p *contained in* M. *The set of all interior points of* M *is called the* interior *of* M, *denoted* $i(M)$.
- p *is said to be an* exterior point *of* M *if there is a neighbourhood* N *of* p *that contains no point of* M. *The set of all exterior points of* M *is called the* exterior *of* M, *denoted* $e(M)$.
- p *is said to be a* boundary point *of* M *if every neighbourhood* N *of* p *contains at least one point in* M *and one point not in* M. *The set of all boundary points of* M *is called the* boundary *of* M, *denoted* $b(M)$.
- *The* closure *of* M *is the smallest closed set which contains* M, *i.e., the union of* M *with the boundary of* M.
- *For any arbitrarily given point* $p \in \mathcal{U}$, *the family of all neighbourhoods of* p *in* $\mathcal{U}$ *will be called the* neighbourhood system *of* p *in* $\mathcal{U}$.

A neighbourhood system $\mathcal{N}$ has the property that every finite intersection of members of $\mathcal{N}$ belongs to $\mathcal{N}$. It is possible to use any topological space which is a model for the RCC axioms. Gotts [10] has shown that every regular connected topological space is a model for the RCC axioms (see also Section 7). So, whenever we refer to a topological space in the remainder of the paper, we mean a regular connected topological space.

Relation	*Model Constraints*	*Entailment Constraints*
$\mathsf{DC}(X,Y)$	$\neg(X \wedge Y)$	$\neg X, \neg Y$
$\mathsf{EC}(X,Y)$	$\neg(\mathbf{I}X \wedge \mathbf{I}Y)$	$\neg(X \wedge Y), \neg X, \neg Y$
$\mathsf{PO}(X,Y)$	—	$\neg(\mathbf{I}X \wedge \mathbf{I}Y), X \to Y, Y \to X, \neg X, \neg Y$
$\mathsf{TPP}(X,Y)$	$X \to Y$	$X \to \mathbf{I}Y, Y \to X, \neg X, \neg Y$
$\mathsf{TPP}^{-1}(X,Y)$	$Y \to X$	$Y \to \mathbf{I}X, X \to Y, \neg X, \neg Y$
$\mathsf{NTPP}(X,Y)$	$X \to \mathbf{I}Y$	$Y \to X, \neg X, \neg Y$
$\mathsf{NTPP}^{-1}(X,Y)$	$Y \to \mathbf{I}X$	$X \to Y, \neg X, \neg Y$
$\mathsf{EQ}(X,Y)$	$X \to Y, Y \to X$	$\neg X, \neg Y$

Table 1: Modal encoding of the eight base relations [2].

3 Modal Encoding & Canonical Models

In this section we will introduce the modal encoding of RCC-8 and a canonical model for this encoding. An introduction to modal logics is given in Appendix A.

3.1 Modal Encoding of RCC8

The modal encoding of RCC-8 was introduced by Bennett [2] and extended in [19]. In both cases the encoding is restricted to regular closed regions. The encoding is based on a set of model and entailment constraints for each base relation, where model constraints must be true and entailment constraints must not be true. Bennett encoded these constraints in modal logic by introducing an S4-operator $\mathbf{I}$ which he interpreted as an interior operator [2]. Table 1 displays these constraints for the eight base relations. In order to combine these constraints to a single modal formula, Bennett introduced a strong S5-operator $\Box$, where $\Box\varphi$ is written for every model constraint φ and $\neg\Box\psi$ for every entailment constraint ψ [2]. All constraints of a single base relations are then combined conjunctively to a single modal formula. In order to represent unions of base relations, the modal formulas of the corresponding base relations are combined disjunctively. In this way every spatial formula $R(X,Y)$ can be transformed to a modal formula $m_1(R(X,Y))$. Two additional constraints $m_2(X)$ are necessary to guarantee that only regular closed regions X are used [19]. A closed region is the closure of an open region and the complement of a closed region is an open region:

$$m_2(X) = \Box(X \leftrightarrow \neg\mathbf{I}\neg\mathbf{I}X) \wedge \Box(\neg X \leftrightarrow \mathbf{I}\neg X).$$

So any set of spatial formulas Θ can be written as a single modal formula $m(\Theta)$

$$m(\Theta) = \left(\bigwedge_{R(X,Y)\in\Theta} m_1(R(X,Y)) \right) \wedge \left(\bigwedge_{X\in Reg(\Theta)} m_2(X) \right),$$

where $Reg(\Theta)$ is the set of spatial variables of Θ. One possible way to interpret modal logics are Kripke semantics. Kripke models of $m(\Theta)$ are based on a set of worlds W and accessibility relations $R_\Box, R_\mathbf{I} \subseteq W \times W$ corresponding to the modal operators $\Box$ and $\mathbf{I}$.

3.2 A Canonical Model of RCC8

A canonical model of a calculus is a structure that allows to model any consistent sentence of the calculus. An obvious canonical model of RCC-8 is the topological space, as every spatial region can be modeled by a subset of the topological space. As described above, the modal encoding of RCC-8 can be interpreted by Kripke models. As the modal encoding of RCC-8 is equivalent to a set of RCC-8 formulas, a canonical model of RCC-8 is a structure that allows a Kripke model for any modal formula obtained by the modal encoding of RCC-8. In order to obtain a canonical model we distinguish different levels of worlds of W [19]. A *world of level 0* is a world which cannot be accessed from any other world with $R_\mathbf{I}$. A *world of level l* is a world which can be accessed with $R_\mathbf{I}$ from a world of level $l-1$ but not from other worlds with a higher level than $l-1$. Based on this hierarchy of worlds, we will define the canonical model of RCC-8.

Definition 3.1 *An* RCC-8-structure $\mathcal{S}_{RCC8} = \langle W, \{R_\Box, R_\mathbf{I}\}, \pi \rangle$ *has the following properties:*

1. *There are only worlds of level* 0 *and* 1.
2. *For every world u of level* 0 *there are exactly* $2n$ *worlds v of level* 1 *with* $uR_\mathbf{I}v$.
3. *For every world v of level* 1 *there is only one world u of level* 0 *with* $uR_\mathbf{I}v$.

$\mathcal{S}_{RCC8}$ contains worlds with all possible validations with respect to $R_\Box$ and $R_\mathbf{I}$. An RCC-8-model $\mathcal{M}$ *of $m(\Theta)$ is a finite subset of $\mathcal{S}_{RCC8}$. In a* polynomial RCC-8-model *the number of worlds is polynomially bounded by the number of regions n.*

In [19] it was proven that if $m(\Theta)$ is satisfiable, there is a polynomial RCC-8-model $\mathcal{M}$ with $\mathcal{M}, w \Vdash m(\Theta)$ that uses $O(n^2)$ different worlds of level 0. So

the RCC-8-structure $\mathcal{S}_{RCC8}$ is a canonical model of RCC-8. More specifically it is a canonical model for the modal encoding of RCC-8. In order to obtain a "topological" canonical model for the topological calculus RCC-8, we give in the next section a topological interpretation of the canonical model.

4 Topological Interpretation of the Canonical Model

The modal encoding of RCC-8 was obtained by introducing a modal operator $\mathbf{I}$ corresponding to the topological interior operator and transferring the topological properties and axioms to modal logic. Using the intended interpretation of $\mathbf{I}$ as an interior operator, it is unclear how the Kripke models we consider, especially the accessibility relations $R_\Box$ and $R_\mathbf{I}$, can be topologically interpreted. In this section we present a way of topologically interpreting RCC-8-models such that all parts of the models can be interpreted consistently on a topological level. The $\mathbf{I}$-operator will not be interpreted as an interior operator, but we will prove that it suffices the intended interpretation of an interior operator.

Lemma 4.1 *Let $\mathcal{M}$ be an RCC-8-model of Θ. Then for every world w all regions of Θ can be divided in three sets $\mathcal{X}_w$, $\mathcal{Y}_w$, and $\mathcal{Z}_w$ such that $\mathcal{M}, w \models \mathbf{I}X_i$, $\mathcal{M}, w \models \mathbf{I}\neg Y_i$, and $\mathcal{M}, w \models Z_i \wedge \neg\mathbf{I}Z_i$ for all $X_i \in \mathcal{X}_w$, $Y_i \in \mathcal{Y}_w$, and $Z_i \in \mathcal{Z}_w$. Then the following relations between regions A and B are impossible, which has some topological consequences:*

Set of A	*Set of B*	*Impossible relations*	*Consequences*
$\mathcal{X}_w$	$\mathcal{X}_w$	DC, EC	$i(A) \cap i(B) \neq \emptyset$
$\mathcal{X}_w$	$\mathcal{Y}_w$	TPP, NTPP, EQ	$i(A) \cap e(B) \neq \emptyset$
$\mathcal{X}_w$	$\mathcal{Z}_w$	DC, EC, TPP, NTPP, EQ	$i(A) \cap b(B) \neq \emptyset$
$\mathcal{Y}_w$	$\mathcal{Y}_w$	–	–
$\mathcal{Y}_w$	$\mathcal{Z}_w$	TPP^{-1}, NTPP^{-1}, EQ	$e(A) \cap b(B) \neq \emptyset$
$\mathcal{Z}_w$	$\mathcal{Z}_w$	DC, NTPP, NTPP^{-1}	$b(A) \cap b(B) \neq \emptyset$[1]

Proof: For every region X and every world w, if X is false in w it is also false in all $R_\mathbf{I}$-successors of w. If X is true in w then it is either true in some or in all $R_\mathbf{I}$-successors of w. This is because of $m_2(\Theta)$. So X must belong to either $\mathcal{X}_w$, $\mathcal{Y}_w$, or $\mathcal{Z}_w$. Most entries in the table follow immediately from the encoding of the

[1] Actually this is not necessarily the case for PO(A, B) if A or B are not internally connected, but assuming this does not contradict any constraint since RCC-8 is not expressive enough to distinguish different kinds of partial overlap. However, it will turn out later that PO is never used in this respect.

relations in modal logic. The only more difficult entry is the relation $\mathsf{EC}(A, B)$ in the third line of the table. This relation is not possible because of the property $\Box(B \rightarrow \neg\mathbf{I}\neg\mathbf{I}B)$ which states that for any world w holding B there is a world v with $wR_{\mathbf{I}}v$ that holds $\mathbf{I}B$. As v also holds $\mathbf{I}A$, the model constraint of $\mathsf{EC}(A, B)$ is violated, so this relation is not possible. □

Theorem 4.2 *Let Θ be a consistent set of spatial formulas, $m(\Theta)$ be the modal encoding of Θ, and $\mathcal{M} = \langle W, \{R_\Box, R_{\mathbf{I}}\}, \pi\rangle$ be an* RCC-8*-model of $m(\Theta)$. Then there is a function $p : W \mapsto \mathcal{U}$ that assigns each world $w \in W$ a point $p(w) \in \mathcal{U}$ and a function $N : W \mapsto 2^{\mathcal{U}}$ that assigns each world $w \in W$ a neighbourhood $N(w)$ of $p(w)$ such that the following properties are satisfied:*[2]

- $p(w) \in X$ *if* $\pi(w, X) = true$,
- $p(w) \notin X$ *if* $\pi(w, X) = false$,
- $p(u) \in N(w)$ *if* $(w, u) \in R_{\mathbf{I}}$.

Proof: All points can be distinguished in interior points, exterior points, and boundary points of regions. If a point is interior or exterior point of a region, there is a neighbourhood of the point such that all points of the neighbourhood are also inside or outside the region, respectively. If a point is a boundary point of a region, every neighbourhood contains points inside the region and points outside the region.

Suppose that the world w holds the modal formulas for the regions X_i, Y_j, and Z_k for all i, j, k as specified in Lemma 4.1. We assume that there is at least one point in the topological space that is in the interior of every $X_i \in \mathcal{X}_w$, in the exterior of every $Y_j \in \mathcal{Y}_w$, and on the boundary of every $Z_k \in \mathcal{Z}_w$ simultaneously. This does not violate any of the topological consequences of Lemma 4.1. We further assume that p maps w to one of these points.

As $p(w)$ is an interior point of X_i, there must be a neighbourhood $N_{X_i}(w) \subset X_i$ where all points corresponding to the $R_{\mathbf{I}}$-successors of w can be placed in. $p(w)$ is an exterior point of Y_j, so there must be a neighbourhood $N_{Y_j}(w)$ disjoint with Y_j where all points corresponding to the $R_{\mathbf{I}}$-successors of w can be placed in. $p(w)$ is a boundary point of Z_k, so any neighbourhood $N_{Z_k}(w)$ has points inside Z_k and outside Z_k where the points corresponding to the $R_{\mathbf{I}}$-successors of w can be placed in. For all regions X_i, Y_j, Z_k the corresponding neighbourhoods are members of the neighbourhood system of $p(w)$, so the intersection $N(w)$ of all these neighbourhoods is also a neighbourhood of $p(w)$ where all points corresponding to the $R_{\mathbf{I}}$-successors of w can be placed in. □

[2] We omit the properties for the accessibility relation $R_\Box$ which would be $p(u) \in \mathcal{U}$ if $(w, u) \in R_\Box$ and $p(w) \in \mathcal{U}$ as we already defined N and p such that only points of $\mathcal{U}$ are used.

Using the above defined functions p and N, we can now transform modal formulas stepwise to topological formulas as follows:

$$\begin{aligned}
\mathcal{M}, w \Vdash \Box\varphi &\mapsto \forall u : p(u) \in \mathcal{U}.\mathcal{M}, u \Vdash \varphi \\
\mathcal{M}, w \nVdash \Box\varphi &\mapsto \exists u : p(u) \in \mathcal{U}.\mathcal{M}, u \nVdash \varphi \\
\mathcal{M}, w \Vdash \mathbf{I}\varphi &\mapsto \forall u : p(u) \in N(w).\mathcal{M}, u \Vdash \varphi \\
\mathcal{M}, w \nVdash \mathbf{I}\varphi &\mapsto \exists u : p(u) \in N(w).\mathcal{M}, u \nVdash \varphi \\
\mathcal{M}, w \Vdash X &\mapsto p(w) \in X \\
\mathcal{M}, w \nVdash X &\mapsto p(w) \notin X
\end{aligned}$$

Therefore $\mathcal{M}, w \Vdash \mathbf{I}X$ can be interpreted as "there is a neighbourhood $N(w)$ of $p(w)$ such that all points of $N(w)$ are in X". This obeys the intended interpretation of $\mathbf{I}$ as an interior operator, as $\mathcal{M}, w \Vdash X$ means that $p(w)$ is in X and $\mathcal{M}, w \Vdash \mathbf{I}X$ means that $p(w)$ is in the interior of X.

In order to make the following argumentation easier, a world mapped to a interior point of X is denoted *interior world* of X, a world mapped to an exterior point of X *exterior world* of X, and a world mapped to a boundary point of X *boundary world* of X. In particular, with the mapping of Theorem 4.2, a world w with $\mathcal{M}, w \Vdash \mathbf{I}X$ is an interior world of X, a world w with $\mathcal{M}, w \Vdash \mathbf{I}\neg X$ is an exterior world of X, and a world w with $\mathcal{M}, w \Vdash X \wedge \neg\mathbf{I}X$ is a boundary world of X.

Lemma 4.3 *Let p and N be functions as determined in Theorem 4.2. Then N can be replaced by any function $N' : W \mapsto 2^{\mathcal{U}}$, with $N'(w) \subseteq N(w)$ for all $w \in W$ if $N'(w)$ is a member of the neighbourhood system of $p(w)$. In particular all neighbourhoods can be regarded as d-dimensional spheres. p has to be changed accordingly.*

Proof: If $p(w)$ is an interior point of X, i.e., $N(w) \subseteq X$, then also $N'(w) \subseteq X$. If $p(w)$ is an exterior point of Y, i.e., $N(w) \cap Y = \emptyset$, then also $N'(w) \cap Y = \emptyset$. If $p(w)$ is a boundary point of Z, i.e., $p(w) \in b(Z)$, $N(w) \cap Z \neq \emptyset$, and $N(w) \not\subseteq Z$ then this is also true for $N'(w)$ under the given conditions. □

5 Models and the Dimension of Space

Based on the RCC-8-models introduced in Section 3 and their topological interpretation, we will first prove that for any RCC-8-model there is a realization in two-dimensional space. It is sufficient to prove this only for sets of base relations as every realization of Θ uses only base relations.[3] For this proof it is important to keep in mind that regions do not have to be internally connected, i.e.,

[3]The relation EQ can be omitted as any pair of spatial variables X and Y with $\mathsf{EQ}(X, Y)$ can be combined to a single spatial variable.

they might consist of different disconnected parts. At the end of this section we will specialize the result to one-dimensional realizations and also generalize it to any other dimension. For three- and higher-dimensional space we will prove that every consistent set Θ can also be realized with internally connected regions.

For the following examination we will restrict regions to be sets of d-dimensional polytopes. Sets are required since regions might consist of several disconnected parts where each part is a single polytope. This restriction will be lifted later and the results can be generalized to arbitrary regular regions. In the two-dimensional case regions are sets of polygons and neighbourhoods are discs according to Lemma 4.3.

Let Θ be a consistent set of spatial formulas and $\mathcal{M}$ be an RCC-8-model of $m(\Theta)$, the modal encoding of Θ. Suppose that only two-dimensional regions are permitted, i.e., the topological space is a two-dimensional plane $\mathcal{U}$. All worlds of $\mathcal{M}$ are mapped to points of $\mathcal{U}$ as specified in Theorem 4.2. The general intuition behind the proof is that all worlds of level 0 together with their $R_{\mathbf{I}}$-successors are mapped to independent neighbourhoods such that each neighbourhood can be placed on an arbitrary but distinct position on the plane. All spatial relations between regions must only hold within the (to be specified) "closure" of these neighbourhoods, the rest of the plane is not important. This is of course only possible when regions are allowed to consist of different parts.

We define a *region vector* $r_i^w = (r_{i,1}^w, \ldots, r_{i,2n}^w)$ for each region X_i and each world w of level 0 that represents the affiliation of the $2n$ $R_{\mathbf{I}}$-successor of w to X_i, i.e., $r_{i,j}^w = 1$ if $\mathcal{M}, v_j \Vdash X_i$ and $r_{i,j}^w = 0$ if $\mathcal{M}, v_j \nVdash X_i$ where v_j is the jth $R_{\mathbf{I}}$-successor of w. Since in the two-dimensional case the neighbourhood $N(w)$ can be chosen as a disc, we suppose that the points $p(v_j)$ corresponding to the $R_{\mathbf{I}}$-successors v_j of w are clockwisely ordered within the disc according to j. If $p(w)$ is not a boundary point of X_i, all values of r_i^w are either 0 (if $p(w)$ is not contained in X_i) or 1 (if $p(w)$ is contained in X_i). If $p(w)$ is a boundary point of X_i, some values of r_i^w are 1 and some are 0.

Lemma 5.1 *If there is a permutation P_w of the values of r_i^w for all worlds w of level 0 such that $(r_{i,P_w(1)}^w, \ldots, r_{i,P_w(2n)}^w)$ is a bitonic sequence for all $1 \leq i \leq n$, then the neighbourhoods $N(w)$ can be placed in a two-dimensional plane such that all spatial relations are satisfied within the neighbourhoods.*

Proof: If r_i^w is a bitonic sequence, i.e., the values of r_i^w are in a form $0^e1^f0^g$ or $1^e0^f1^g$ for $e, f, g \geq 0$, and $p(w)$ is a boundary point of X_i then the mappings of the worlds of level 1 corresponding to the values in r_i^w can be separated into points inside X_i and points outside X_i by at most two line segments meeting at $p(w)$ (see Figure 2). These line segments can be regarded as the part of the boundary of X_i which is inside $N(w)$. So, neighbourhoods can be separated in an interior

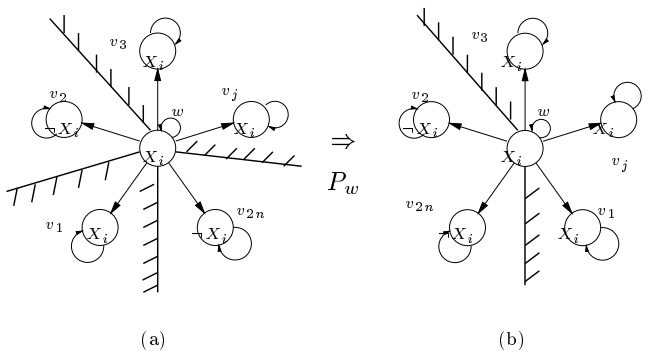

Figure 2: Permutation P_w of the $R_{\mathbf{I}}$-successors of a world w.

and an exterior part of a region by a one-dimensional boundary. Therefore all neighbourhoods can be placed in a two-dimensional plane. As the permutation of the $R_{\mathbf{I}}$-successors has no influence on the relations between the regions, all spatial relations between the regions hold within the neighbourhoods. □

Actually a permutation as described in the previous Lemma is not necessary to guarantee two-dimensional realizations. A region might look as shown in Figure 2(a), but in this case we restrict the shape and the internal connection of the regions by the neighbourhoods we are using which is not at all desirable. In contrast to this, the shape and the internal connection of the regions is not restricted by the neighbourhoods when it it possible to find a permutation as described in Lemma 5.1. Indeed, it is necessary for one-dimensional realizations and realizations with internally connected regions.

In order to find out whether there always exists a permutation P_w, we have to take a closer look at how boundary worlds are introduced and how worlds can be forced to be boundary worlds, since a permutation is only necessary for boundary worlds. As a world w of level 0 is mapped to a boundary point of X if $w \Vdash X$ and $v \nVdash X$ for a world v with $wR_{\mathbf{I}}v$, we have to find out which of the constraints force $w \Vdash X$ if $v \nVdash X$ or force $v \nVdash X$ if $w \Vdash X$, respectively.

Proposition 5.2 *Boundary worlds are introduced only by the following relations:*

$\mathsf{EC}(X,Y)$: $\neg\Box(\neg(X \wedge Y))$ *introduces a world which is a boundary world of* X *and* Y *because of* $\Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y))$.

$\mathsf{TPP}(X,Y)$: $\neg\Box(X \to \mathbf{I}Y)$ *introduces a world which is a boundary world of* X *and* Y *because of* $\Box(X \to Y)$.

$\mathsf{TPP}^{-1}(X,Y)$: $\neg\Box(Y \to \mathbf{I}X)$ *introduces a world which is boundary world of* X *and* Y *because of* $\Box(Y \to X)$.

Apart from the above worlds that are introduced as boundary worlds of particular regions, worlds can also be forced to be boundary worlds of other regions.

Proposition 5.3 *A world can be forced to be a boundary world of X and Y only with the following constraints:*

$\Box(X \to Y)$: – *If w is a boundary world of Y and X is true in w, then w must also be a boundary world of X.*
– *If w is a boundary world of X and $\neg Y$ is true in an $R_{\mathbf{I}}$-successor of w, then w is also a boundary world of Y.*

$\Box(Y \to X)$: – *If w is a boundary world of X and Y is true in w, then w is also a boundary world of Y.*
– *If w is a boundary world of Y and $\neg X$ is true in an $R_{\mathbf{I}}$-successor of w, then w is also a boundary world of X.*

$\Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y))$: – *If X and Y are true in w, then w is a boundary world of both X and Y.*

For the constraints $\Box(X \to Y)$ and $\Box(Y \to X)$ w must already be a boundary world of some other region so w must be introduced by one of the relations $\mathsf{EC}(X,Y)$, $\mathsf{TPP}(X,Y)$, or $\mathsf{TPP}^{-1}(X,Y)$. If w is forced to be a boundary world of X and Y with the constraint $\Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y))$, then X and Y must be true in w. This can only be forced when there is a region Z_1 with $\mathsf{TPP}(Z_1, X)$ and Z_1 is true in w and a region Z_2 with $\mathsf{TPP}(Z_2, Y)$ and Z_2 is true in w. This is only true when w is a boundary world of Z_1 and Z_2 introduced by $\mathsf{EC}(Z_1, Z_2)$. So in any case when a world is forced to be a boundary world of some region it must already be a boundary world of other regions introduced as described in Proposition 5.2.

We will now have a look at how regions must be related in order to force a world to be a boundary world of these regions. Suppose that w is a boundary world of X and Y, introduced by either $\mathsf{EC}(X,Y)$ or $\mathsf{TPP}(X,Y)$.[4] We will write $X|Y$ in order to express that we can either use X or Y but always the same. With one of the following constraints it can be forced that w is also a boundary world of $Z_i \neq X, Y$:

$\Box(Z_i \to (X|Y))$: $\mathsf{TPP}(Z_i, X|Y)$ and $w \mid\!\vdash Z_i$
$\Box(\neg(\mathbf{I}Z \wedge \mathbf{I}(X|Y)))$: $\mathsf{EC}(Z_i, X|Y)$ and $w \mid\!\vdash Z_i$

$w \mid\!\vdash Z_i$ can be forced with one of the following constraints:

[4] We omit $\mathsf{TPP}^{-1}(X,Y)$ as $\mathsf{TPP}^{-1}(X,Y) = \mathsf{TPP}(Y,X)$ and the order is not important.

$\Box(U \to Z_i)$: $\mathsf{TPP}(U, Z_i)$ and $w \Vdash U$
$\Box(U \to \mathbf{I}Z_i)$: $\mathsf{NTPP}(U, Z_i)$ and $w \Vdash U$

When we compose these relations, we obtain the relation between U and $X|Y$.

$$\begin{aligned}
\mathsf{TPP}(U, Z_i) \circ \mathsf{TPP}(Z_i, X|Y) &= \mathsf{TPP}, \mathsf{NTPP}(U, X|Y) \\
\mathsf{TPP}(U, Z_i) \circ \mathsf{EC}(Z_i, X|Y) &= \mathsf{DC}, \mathsf{EC}(U, X|Y) \\
\mathsf{NTPP}(U, Z_i) \circ \mathsf{TPP}(Z_i, X|Y) &= \mathsf{NTPP}(U, X|Y) \\
\mathsf{NTPP}(U, Z_i) \circ \mathsf{EC}(Z_i, X|Y) &= \mathsf{DC}(U, X|Y)
\end{aligned}$$

As w is a boundary world of X and Y, $\mathsf{DC}(U, X|Y)$ and $\mathsf{NTPP}(U, X|Y)$ are not possible together with $w \Vdash U$, so $\mathsf{TPP}(U, Z_i)$ must hold. In order to force $w \Vdash U$, U must either be equal to X or Y or there must be a world U_2 with $\mathsf{TPP}(U_2, U)$ and so on until there is a world U_m that is equal to X or Y, so $\mathsf{TPP}(X, U)$ or $\mathsf{TPP}(Y, U)$ must hold. This results in six different possibilities.

1. $U = X$ or $U = Y$:
   a. $\mathsf{TPP}(X, Y), \mathsf{TPP}(X, Z_i)$, and $\mathsf{TPP}(Z_i, Y)$ hold.
   b. $\mathsf{EC}(X, Y), \mathsf{TPP}(X, Z_i)$, and $\mathsf{EC}(Z_i, Y)$ hold.
   c. $\mathsf{EC}(X, Y), \mathsf{TPP}(Y, Z_i)$ and $\mathsf{EC}(Z_i, X)$ hold.
2. $\mathsf{TPP}(X, U)$ or $\mathsf{TPP}(Y, U)$:
   a. $\mathsf{TPP}(X, Y), \mathsf{TPP}(X, U), \mathsf{TPP}(X, Z_i)$, and $\mathsf{TPP}(Z_i, Y)$ hold.
   b. $\mathsf{EC}(X, Y), \mathsf{TPP}(X, U), \mathsf{TPP}(X, Z_i)$, and $\mathsf{EC}(Z_i, Y)$ hold.
   c. $\mathsf{EC}(X, Y), \mathsf{TPP}(Y, U), \mathsf{TPP}(Y, Z_i)$, and $\mathsf{EC}(Z_i, X)$ hold.

In any case where w is forced to be a boundary point of a region Z_i the relations between Z_i and X and Y are either $\mathsf{TPP}(X, Z_i)$ and $\mathsf{TPP}(Z_i, Y)$ if $\mathsf{TPP}(X, Y)$ holds or $\mathsf{TPP}(X|Y, Z_i)$ and $\mathsf{EC}(Z_i, Y|X)$ if $\mathsf{EC}(X, Y)$ holds. As the different regions Z_i, Z_j all have the boundary world w in common, only the relations $\mathsf{EC}, \mathsf{PO}, \mathsf{TPP}$, or TPP^{-1} can hold between them.

So we have to distinguish two cases, $\mathsf{EC}(X, Y)$ and $\mathsf{TPP}(X, Y)$. For $\mathsf{EC}(X, Y)$ some of the $R_{\mathbf{I}}$-successors of level 1 of w are in X but not in Y and some are in Y but not in X, the others are neither in X nor in Y. For all regions Z_i with $\mathsf{TPP}(X, Z_i)$ and $\mathsf{EC}(Z_i, Y)$ all $R_{\mathbf{I}}$-successors of w hold Z_i if they hold X and the same for all regions Z_j with $\mathsf{TPP}(Y, Z_j)$ and $\mathsf{EC}(Z_j, X)$, so all $R_{\mathbf{I}}$-successors contained in X force the same formulas and all $R_{\mathbf{I}}$-successors contained in Y force the same formulas. The $R_{\mathbf{I}}$-successors v of w which are not in X or Y are

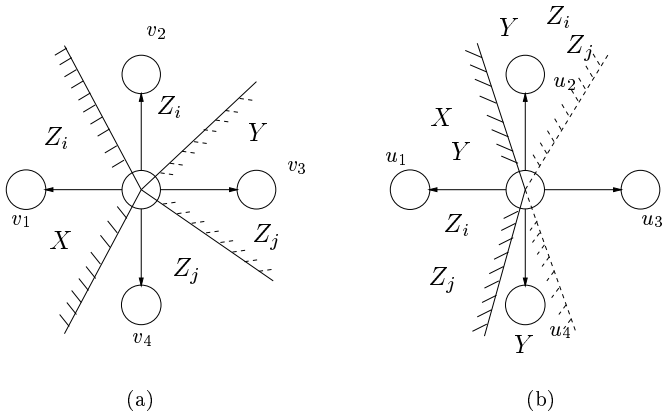

Figure 3: The four sufficient types of worlds for $\mathsf{EC}(X, Y)$ (a) and $\mathsf{TPP}(X, Y)$ (b). The solid line is the boundary of X, the dotted line the boundary of Y where the hatched areas indicate the interior of X and Y.

a kind of "don't care" worlds, as the only two requirements for these worlds are that Z_k is true in v whenever $Z_{k'}$ is true in v if $\mathsf{TPP}(Z_{k'}, Z_k)$ holds and that Z_k and $Z_{k'}$ are not both true in v if $\mathsf{EC}(Z_k, Z_{k'})$ holds. Everything else is arbitrary without violating any constraint. Therefore we assume that for some worlds v all Z_i with $\mathsf{TPP}(X, Z_i)$ are true which are false for the worlds where all Z_j with $\mathsf{TPP}(Y, Z_j)$ are true and *vice-versa*. It can be seen that there are four different types of worlds v_1, v_2, v_3, v_4 with the following properties (see Figure 3(a)):

1. $v_1 \Vdash X \wedge Z_i$, $v_1 \nVdash Y$ and $v_1 \nVdash Z_j$.
2. $v_2 \Vdash Z_i$, $v_2 \nVdash X$, $v_2 \nVdash Y$ and $v_2 \nVdash Z_j$.
3. $v_3 \Vdash Y \wedge Z_j$, $v_3 \nVdash X$ and $v_3 \nVdash Z_i$.
4. $v_4 \Vdash Z_j$, $v_4 \nVdash X$, $v_4 \nVdash Y$ and $v_4 \nVdash Z_i$.

It is obvious that when the worlds are ordered like this, r^w is a bitonic sequence for all regions.

For $\mathsf{TPP}(X, Y)$ all $R_{\mathbf{I}}$-successors of w that are in X are also in Y, some $R_{\mathbf{I}}$-successors are neither in X nor in Y, and some are in Y but not in X. The latter worlds u are again a kind of "don't care" worlds, as the only requirement for these worlds is that Z_i is true whenever Z_j is true in these worlds for any two regions Z_i, Z_j with $\mathsf{TPP}(X, Z_i), \mathsf{TPP}(Z_j, Y)$ and $\mathsf{TPP}(Z_i, Z_j)$[5]. Everything else is again arbitrary without violating any constraint. Therefore we assume that for

[5]If $\mathsf{PO}(Z_i, Z_j)$ holds, then there is no additional requirement on these worlds.

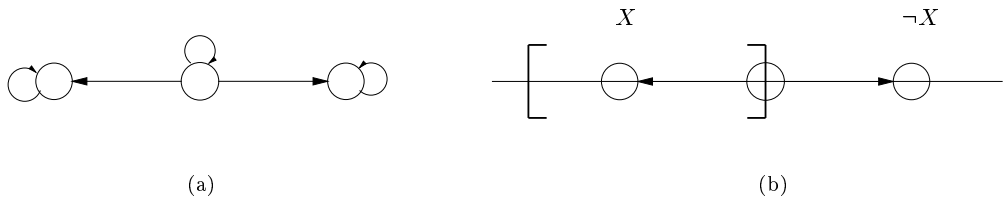

Figure 4: (a) shows the structure of the new canonical model, the reduced RCC-8-structure. In (b) it is shown how the neighbourhoods can be placed in one-dimensional space. The two brackets indicate a possible one-dimensional region X where the neighbourhood defines a boundary point of X.

all these worlds either both Z_i and Z_j are true in u or both are false in u. Then we have four different types of worlds u_1, u_2, u_3, u_4 with the following properties (see Figure 3(b)):

1. $u_1 \Vdash X \wedge Y \wedge Z_i \wedge Z_j$
2. $u_2 \Vdash Y \wedge Z_i \wedge Z_j$, and $u_2 \nVdash X$.
3. $u_3 \nVdash X$, $u_3 \nVdash Y$, $u_3 \nVdash Z_i$, and $u_3 \nVdash Z_j$.
4. $u_4 \Vdash Y$, $u_4 \nVdash X$, $u_4 \nVdash Z_i$, and $u_4 \nVdash Z_j$.

Also in this case it is obvious that when the worlds are ordered like this, r^w is a bitonic sequence for all regions. We have shown that for any consistent set of spatial formulas we can find a permutation of the worlds of level 1 such that r^w is a bitonic sequence for all regions.

Lemma 5.4 *There is always a permutation P_w of the values of r_i^w for all worlds w of level 0 such that $(r^w_{i,P_w(1)}, \ldots, r^w_{i,P_w(2n)})$ is a bitonic sequence for all $1 \leq i \leq n$.*

The number of different worlds can be reduced even more as the "don't care" worlds u_2, u_4 and v_2, v_4 are not necessary. There is no constraint that forces the existence of an $R_{\mathbf{I}}$-successor of a boundary world of X and Y where X and Y is false and Z_i or Z_j must be true or where Y is true and X false when $\mathsf{TPP}(X, Y)$. So every world of level 0 needs only two different $R_{\mathbf{I}}$-successors. This leads to a very simple canonical model shown in Figure 4(a). We call this *reduced* RCC-8-*structure* and the corresponding models *reduced* RCC-8-*models*.

We can now apply Lemma 5.1 and place all neighbourhoods independently on the plane while all relations between spatial regions hold within the neighbourhoods. As the neighbourhoods are open sets, it is also possible that two regions which

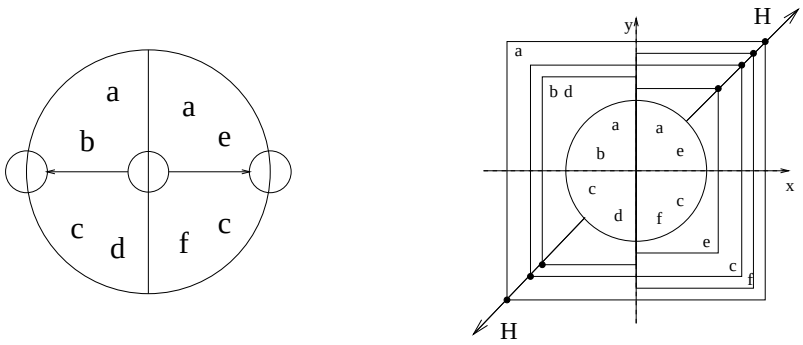

Figure 5: (a) shows the two-dimensional neighbourhood of a boundary point which is divided by the boundary. In (b) the neighbourhood is closed wrt. the hierarchy H_Θ of the affiliated regions.

are not allowed to have a common boundary point are equal within a particular neighbourhood. In order to obtain regions from the neighbourhoods we have to *close* every neighbourhood, i.e., for every neighbourhood $N(w)$ find the closure of the part of every region which is affiliated with $N(w)$. For this, both sides of every neighbourhood (see Figure 5a) can be treated almost independently. All regions which are affiliated with the same side of a neighbourhood are either overlapping or one is part of the other, i.e., TPP or NTPP. For the closure of the neighbourhoods all "part of" relations must be fulfilled, whereas the partial overlap relation is not important.

In order to fulfil the "part of" relations, we have to find a *hierarchy* H_Θ of the regions such that such that a "part of" b iff $H_\Theta(a) < H_\Theta(b)$.[6] The parts of all regions affiliated with a neighbourhood can then be closed as rectangles according to the hierarchy H_Θ, i.e., regions of the same level are equal (for a particular neighbourhood) and are part of all regions of a higher level (see Figure 5b). A neighbourhood can be closed in any higher dimension d. The hierarchy of regions is then measured along the diagonal of the d-dimensional hypercube.

Lemma 5.5 *If a set of spatial formulas Θ is consistent, there is a consistent instantiation of all spatial variables with two-dimensional regions. This is also true when regions are (sets of) polygons.*

[6] This corresponds to determining the finish time of depth-first search of every node of a graph G_Θ where all regions are vertices V_Θ and all "part of" relations are directed edges E_Θ. Since any cycle in the graph leads to an inconsistency, we have a directed acyclic graph and the finish times can always be found in time $O(V_\Theta + E_\Theta)$ [6]

As it is always possible to align the three points corresponding to a world w of level 0 and its two $R_{\mathbf{I}}$-successors of level 1, the neighbourhood $N(w)$ of $p(w)$ can be placed on a line, i.e., even a one-dimensional neighbourhood is sufficient for every point (see Figure 4b). All neighbourhoods can be placed independently on a single line and closed as intervals according to the hierarchy H_Θ of the affiliated regions. So if internal connection of regions is not required, regions can be realized consistently even in one dimension. As a neighbourhood can be closed in any dimension $d \geq 1$ the following theorem holds.

Theorem 5.6 *If a set of spatial formulas Θ is consistent, there is a consistent instantiation of all spatial variables with regions in any dimension $d \geq 1$ where regions are (sets of) d-dimensional polytopes.*

We can further show that for three- and higher-dimensional space all regions can be realized as internally connected regions. For this we will construct a three dimensional realization of internally connected regions out of a two-dimensional realization of internally disconnected regions. The same construction can be made for any higher dimension.

Theorem 5.7 *If a set of spatial formulas Θ is consistent, there is a consistent instantiation of all spatial variables in any dimension $d \geq 3$ where all regions are internally connected polytopes.*

Proof: Suppose that Θ is consistent. With the following construction we obtain a three-dimensional realization of internally connected regions starting from a two-dimensional realization. Place all (closed) neighbourhoods on a circle in the plane such that the linear boundary corresponding to any neighbourhood can be elongated to the center of the circle (see Figure 6a). Proceed according to the hierarchy H_Θ of regions, by first choosing pairwise distinct intervals on the positive axis of the third dimension (z-axis) for every region of the lowest level in the hierarchy, i.e., for the regions that do not contain any other region. Build pipes for every part of the regions from the plane (z=0) to the endpoint of the corresponding interval. Connect the pipes of every region in the center within the range of the corresponding interval (see Figure 6b). Next the regions of the second level of the hierarchy, i.e., regions that only contain regions of the first level are connected. To do this we choose intervals on the z-axis for all regions of the second level such that the intervals contain all intervals of the contained regions but do not overlap with any other intervals. Build pipes up to the endpoint of the corresponding interval with the largest z-value and connect the pipes of every region in the center within the range of all corresponding intervals (see Figure 6c). This is repeated for every level of the hierarchy until all regions are connected. After that, all the neighbourhoods must be closed on the negative

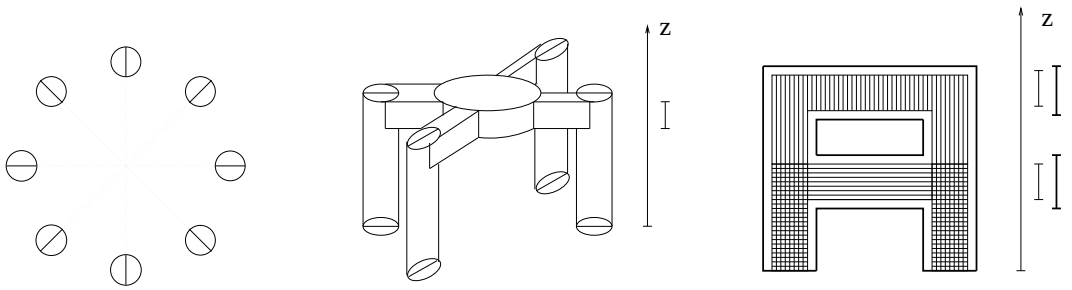

Figure 6: Construction of the three-dimensional realization. (a) shows the placement of the two-dimensional neighbourhoods on a circle, (b) shows how the pieces of a region are connected on a particular level and (c) shows how a region is connected (bold line) that contains the vertical and the horizontal hashed regions.

z-axis according to H_Θ. Obviously, with this construction all regions are internally connected. Furthermore all internally connected three-dimensional regions hold the same base relations as the two-dimensional realizations from which we started the construction. This is because all intervals on the z-axis are either contained in each other or are distinct, they have no common boundary points. All intervals corresponding to region X are contained in the intervals of region Y iff $\mathsf{NTPP}(X, Y)$ or $\mathsf{TPP}(X, Y)$. When two regions are disconnected they remain disconnected as they are not affiliated to the same neighbourhoods. Two externally connected regions remain externally connected because the neighbourhoods were placed on the circle such that their boundary can be elongated to the center. Therefore if two of these regions are both affiliated with the same neighbourhood, their pipes are externally connected and the horizontal connection of the single pipes is distinct. All other requirements of relations as, e.g., a common boundary are already met by the pipes.

With the same construction, a $d + 1$-dimensional realization of internally connected regions can be obtained from a d-dimensional realization of internally disconnected regions. In this case, the (d-dimensional) neighbourhoods must be placed on a d-dimensional sphere and the intervals must be chosen at the $d + 1$ dimensional axis. All constructions kept the polytopic shape of the regions, so every region can be realized as a (d-dimensional) polytope. □

The restriction of regions to be polytopes can immediately be generalized to an arbitrary shape of regions.

6 Applicability of the Canonical Model

With the canonical model every consistent spatial configuration can be modelled, i.e., a set of regions and their relations can be represented as a set of characteristic points and information about their neighbourhood. In the previous section we were reporting about the existence of RCC-8 models, but it is also important to determine these models. Then they can be used, e.g., to generate consistent scenarios.

6.1 Determination of Kripke Models

Given a set of spatial formulas Θ, we have to find a reduced Kripke model for the modal encoding of RCC-8. The reduced Kripke frame of the model, i.e., the worlds and the accessibility relations are already known from the entailment constraints [19], but we have to find a validation for every world and every region. This can be done by transforming the modal formula into a propositional formula with respect to the reduced Kripke frame [19]. As every world of level 0 has now only two $R_{\mathbf{I}}$-successors instead of $2n$, the propositional formula is much smaller than it was in [19]. In return, the propositional formula is not a Horn formula anymore.[7] Furthermore we cannot use any validation, as we have to keep the additional constraint which made the reduced Kripke models possible, namely, that only those worlds are boundary worlds which are forced to be boundary worlds. As it is known which worlds are boundary worlds (see Proposition 5.2), the validation of the boundary world and the two regions which introduced the boundary world can be chosen according to the relation holding between the two regions. The non-boundary worlds do not need any $R_{\mathbf{I}}$-successors as their validation must be equal to the validation of the level 0 world. The rest can be resolved using unit-resolution. Those variables that are not resolved can be set to false. This works even with the non Horn clauses. The only case where this might fail, is when for a particular region X a level 0 boundary world is positive, but nothing was resolved about the two $R_{\mathbf{I}}$-successors. By analyzing the constraints it turns out that this case is impossible, so unit-resolution is sufficient. As there are $O(n^2)$ different worlds and n regions, there are $O(n^4)$ clauses [19], so a Kripke model can be determined in time $O(n^4)$.

[7] This is because of the constraint $\Box(X \rightarrow \neg\mathbf{I}\neg\mathbf{I}X)$ which is transformed to $\bigwedge_{w\in W_0}(\neg X_w \vee X_w^1 \vee X_w^2)$ in the notation of [19].

6.2 Generating a consistent scenario

Suppose we have given a model of a consistent set of RCC-8 formulas Θ. We have to distinguish the tasks of generating a consistent scenario of internally connected and disconnected regions. A consistent scenario of disconnected regions in d-dimensional space can be simply obtained by placing the $O(n^2)$ different neighbourhoods in the d-dimensional space and close each neighbourhood as specified in Section 5 (see also Figure 5). For this the hierarchy of regions has to be known, which can be computed in time $O(n + P_\Theta)$ where P_Θ is the number of "part of" relations in Θ (see Footnote 6). Let $A_\Theta \leq n$ be the maximal number of regions affiliated to a neighbourhood ($A_\Theta \in O(n)$), then the closure of a neighbourhood can be computed in time $O(A_\Theta)$.

Theorem 6.1 *A consistent scenario of a set of* RCC-8 *relations* Θ *can be generated in d-dimensional space* ($d \geq 1$) *in times* $O(n^2 A_\Theta)$ *when the regions are allowed to be disconnected.*

In order to generate a consistent scenario of internally connected regions in d-dimensional space we can use the construction of the proof to Theorem 5.7. For each of the $O(n^2)$ different neighbourhoods and every affiliated region we have to find the intervals. The number of intervals of a particular region a is equal to the number of regions of the lowest level of the hierarchy H_Θ that are contained in a. Let $I_\Theta \leq n$ be the maximal number of regions of the lowest level of H_Θ that are contained in a region ($I_\Theta \in O(n)$).

Theorem 6.2 *A consistent scenario of internally connected regions of a set of* RCC-8 *relations* Θ *can be generated in d-dimensional space in times* $O(n^2 A_\Theta I_\Theta)$.

7 Discussion & Related Work

There is some work on identifying canonical models for the RCC axioms, i.e., determining what mathematical structures fulfil all the RCC axioms, as, e.g., every region has a non-tangential proper part. Gotts [10] found that every connected T_3 space, i.e., every connected and regular topological space is a model for the RCC axioms. Stell and Worboys [20] identified a whole class of models. Both approaches only describe models of the RCC axioms, i.e., what kind of regions can be used at all. When additional constraints expressing relationships between regions are added, these results do not say anything about models anymore.[8] They

[8] Consider the hypothetical case of a set of spatial formulas which is realizable in two- but not in one-dimensional space. Then the one-dimensional space is still a model of the RCC axioms, but not of RCC-8.

are also by no means constructive, as they do not provide a way of effectively representing regions or generating scenarios.

There were some approaches on dealing with dimension and internal connectedness of regions. In particular it was tried to specify predicates and suitable axioms in order to restrict dimension and connectedness of regions [3, 9]. As all regions must have the same dimension anyway, using our results it is not necessary to specify the dimension of regions explicitly if internally disconnected regions are permitted. If internally connected regions are required, these predicates only have an influence on the consistency of a set of spatial relations in one- or two-dimensional applications, as in three- and higher-dimensional space all regions may be either internally connected or disconnected. Forcing internal connectedness of regions in two-dimensional space is a serious problem as there are no complete algorithms for dealing with this task. As Grigni et al. [11] pointed out, a well-known open problem in graph theory, which is NP-hard, but not known to be in NP [14, 15] can be reduced to the consistency problem for two-dimensional internally connected regions.

It is certainly the better approach to have an additional connectedness predicate than forcing all regions to be internally connected (which is done, e.g., by the similar calculus of Egenhofer [7]), as there are many applications where regions are in fact disconnected. Within the area of geographical information systems, e.g., which offer a great variety of possible applications, many countries or other geographical entities are not internally connected regions. In areas like computer vision it is often dealt with two-dimensional projections of the three-dimensional space, where many connected objects are perceived as disconnected objects due to occlusion. This is also true for robot navigation, where the world often consists of a two-dimensional abstraction of the three-dimensional space.

With the result on realizations in any dimension it becomes possible now to use RCC-8 for temporal reasoning tasks, in particular when non-convex intervals are allowed and the direction of the time is not important, as, e.g., in some scheduling problems. This is in contrast to previous approaches that used temporal calculi for spatial reasoning [12].

8 Summary

We identified a canonical model of RCC-8 based on Kripke semantics. In order to obtain a "topological" canonical model, we gave a topological interpretation of the Kripke models such that regions can be represented by points in the topological space and information about the neighbourhood of these points with respect to the spatial relations holding between the regions. Using this canonical model,

we proved that every consistent set of spatial formulas has a realization in any dimension when regions are not forced to be internally connected, which is the case for regions as used by RCC-8. Furthermore, we proved that for three- and higher dimensional space there is always a realization with internally connected regions. Further we give for the first time algorithms for generating consistent scenarios of either internally connected or disconnected regions. Future work includes analyzing the usage of the canonical model for dealing with the special case of two-dimensional internally connected regions as well as analyzing the cognitive meaning of the canonical model [13].

Acknowledgments

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A Basics on Modal Logic

Propositional modal logic [8, 4] has the same syntax as standard propositional logic except for an additional unary operator $\Box$. One common approach to interpret modal logical formulas is the *Kripke semantics*, where models $\mathcal{M}$ are build upon so-called *frames* $\mathcal{F} = \langle W, R\rangle$ that consist of a set of worlds W together with an accessibility relation $R \subseteq W \times W$ defined on worlds, and a validation π that assigns truth values to all the propositional atoms in every world. If a world $v \in W$ is accessible by a world $w \in W$, i.e. $(w, v) \in R$, we say that v is an *R-successor* of w.

The truth of a modal formula ϕ in a world w of a model $\mathcal{M}$, written as $\mathcal{M}, w \Vdash \phi$, is defined on the inductive structure of ϕ:

$\mathcal{M}, w \Vdash a$ for an atom a	iff	$\pi(w, a) =$ true
$\mathcal{M}, w \Vdash \neg\phi$	iff	$\mathcal{M}, w \nVdash \phi$
$\mathcal{M}, w \Vdash \phi \wedge \psi$	iff	$\mathcal{M}, w \Vdash \phi$ and $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \phi \vee \psi$	iff	$\mathcal{M}, w \Vdash \phi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \phi \rightarrow \psi$	iff	$\mathcal{M}, w \nVdash \phi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \Box\phi$	iff	for all u with wRu: $\mathcal{M}, u \Vdash \phi$

It can be seen that a modal operator is closely related to the corresponding accessibility relation. A modal logic is determined by a set of frames, while frames can be determined either by specifying particular relations or axioms that

must hold in all frames. Some well-known modal logics are given in the following table:

Name	Relation	Axiom
K	–	$\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
T	reflexive	$\Box\phi \rightarrow \phi$
4	transitive	$\Box\phi \rightarrow \Box\Box\phi$
5	euclidian (wRu and wRv implies uRv)	$\neg\Box\phi \rightarrow \Box\neg\Box\phi$

K is the set of all frames. Other modal logics can be obtained by combining these logics. Two of them which are of particular interest for the modal encoding of RCC-8 are S4 and S5. S4 are all frames that are K and T as well as 4. S5-frames are both S4 and 5, they have the special property that worlds are clustered. All worlds of a particular cluster are mutually accessible. Bennett [2] calls the S5-frames he uses a *strong* S5-fram, as all worlds are in the same cluster. The modal operator is named according to the modal logic, so, e.g., the operator of an S5-frame is called S5 operator.

In a multi-modal logic it is possible to use more than one modal operator, where every operator $\Box_i$ has it's own accessibility relation R_i, defined on the same set of worlds. The $\Box$-rule defined above has to be changed to different $\Box_i$-rules: $\mathcal{M}, w \Vdash \Box_i\phi$ iff for all u with wR_iu: $\mathcal{M}, u \Vdash \phi$. The other rules stay the same.

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