On the Complexity of Qualitative Spatial Reasoning: A Maximal Tractable Fragment of the Region Connection Calculus*

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Abstract

The computational properties of qualitative spatial reasoning have been investigated to some degree. However, the question for the boundary between polynomial and NP-hard reasoning problems has not been addressed yet. In this paper we explore this boundary in the "Region Connection Calculus" RCC-8. We extend Bennett's encoding of RCC-8 in modal logic. Based on this encoding, we prove that reasoning is NP-complete in general and identify a maximal tractable subset of the relations in RCC-8 that contains all base relations. Further, we show that for this subset path-consistency is sufficient for deciding consistency.

1 Introduction

When describing a spatial configuration or when reasoning about such a configuration, often it is not possible or desirable to obtain precise, quantitative data. In these cases, qualitative reasoning about spatial configurations may be used.

One particular approach in this context has been developed by Randell, Cui, and Cohn [20], the so-called *Region Connection Calculus* (RCC), which is based on binary topological relations. One variant of this calculus, RCC-8, uses eight mutually exhaustive and pairwise disjoint relations, called base relations, to

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describe the topological relationship between two regions (see also Egenhofer [6]).

Some of the computational properties of this calculus have been analyzed by Grigni et al. [11] and Nebel [16]. However, no attempt has yet been made to determine the boundary between polynomial and NP-hard fragments of RCC-8, as it has been done for Allen's [1] interval calculus [18]. We address this problem and identify a maximal fragment of RCC-8 that is still tractable and contains all base relations.

As in the case of qualitative temporal reasoning, this proof relies on a computer generated case-analysis that cannot be reproduced in a research paper.[1] Further, we show that for this fragment path-consistency is sufficient for deciding consistency.

The rest of the paper is structured as follows. In Section 2 we introduce RCC-8 and some basic terminology and definitions that are used in the rest of the paper. In section 3 we introduce and extend Bennett's [3] encoding of RCC-8 in a multi-modal propositional logic. In Section 4, we show that reasoning in RCC-8 is NP-hard by proving that the simpler calculus RCC-5 is already NP-hard. Using the modal encoding of RCC-8, we show in Section 5 how reasoning in RCC-8 can be reduced to satisfiability in propositional logic (SAT). Based on that, in Section 6 a tractable subset of RCC-8 is identified which contains only relations representable as Horn clauses. Further, using a computer generated case-analysis we prove that the set is maximal. In Section 7 we discuss the applicability of the path-consistency algorithm, and in Section 8 we give an estimation of the applicability of the maximal tractable subset to the general reasoning problem. In the appendices, we give a concise introduction to modal logic and an enumeration of the tractable fragment of RCC-8.

2 Qualitative Spatial Reasoning with RCC

RCC is a topological approach to qualitative spatial representation and reasoning where *spatial regions* are subsets of topological space [20]. Relationships between spatial regions are defined in terms of the relation $C(a, b)$ which is true iff the closure of region a is connected to the closure of region b, i.e. if they share a common point. Regions themselves do not have to be internally connected, i.e. a region may consist of different disconnected parts. The domain of *spatial variables* (denoted as X, Y, Z) is the whole topological space. The RCC theory is formulated in first order predicate calculus [20].

In this work we will focus on RCC-8, but most of our results can easily be applied to RCC-5, a subset of RCC-8 [2]. RCC-8 uses a set of eight pairwise disjoint and mutually exhaustive relations, called *base relations*, denoted as DC, EC, PO, EQ, TPP, NTPP, TPP^{-1}, and NTPP^{-1}, with the meaning of *DisConnected, Externally Connected, Partial Overlap, EQual, Tangential Proper Part, Non-*

[1] The programs can be obtained from the authors.

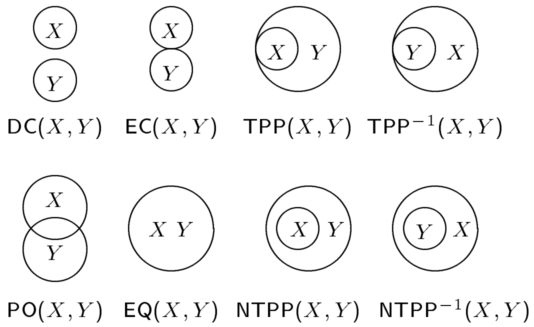

Figure 1: Two-dimensional examples for the eight base relations of RCC-8.

Tangential Proper Part, and their converses. Examples for these relations are shown in Figure 1. Exactly one of these relations holds between any two spatial regions. In RCC-5 the boundary of a region is not taken into account, i.e. one does not distinguish between DC and EC and between TPP and NTPP. These relations are combined to the RCC-5 base relations DR for *DiscRete* and PP for *Proper Part*, respectively. *Spatial formulas* are written as XRY or $R(X, Y)$, where R is a spatial relation.

Sometimes it is not known which of the eight base relations holds between two regions, but it is possible to restrict to some of them. In order to represent this, unions of base relations can be used. Since base relations are pairwise disjoint, this results in 2^8 different RCC-8 relations, including the union of all base relations, which is called *universal relation*. In the following we will write sets of base relations to denote these unions. Using this notation, DR, e.g., is identical to $\{\mathsf{DC}, \mathsf{EC}\}$. Apart from union ($\cup$), other operations are defined, namely, converse ($^{\smile}$), intersection ($\cap$), and composition ($\circ$) of relations. The formal definitions of these operations are:

$$\begin{array}{rrcl}
\forall X, Y: & X(R \cup S)Y & \leftrightarrow & XRY \vee XSY, \\
\forall X, Y: & X(R \cap S)Y & \leftrightarrow & XRY \wedge XSY, \\
\forall X, Y: & XR^{\smile}Y & \leftrightarrow & YRX, \\
\forall X, Y: & X(R \circ S)Y & \leftrightarrow & \exists Z : (XRZ \wedge ZSY).
\end{array}$$

Converse, intersection and union of relations can easily be obtained by performing the corresponding set theoretic operations. Composition of base relations has to be computed using the formal definitions of the relations [19, 2]. The compositions of the eight base relations are shown in Table 1. Every entry in the composition table specifies the relation obtained by composing the base relation of the corresponding row with the base relation of the corresponding column. Composition of two arbitrary RCC-8 relations can be obtained by computing the union of the composition of the base relations.

A *spatial configuration* can be described by a set Θ of spatial formulas. One important computational problem is deciding *consistency* of Θ, i.e. deciding

$\circ$	DC	EC	PO	TPP	NTPP	TPP^{-1}	NTPP^{-1}	EQ
DC	*	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC	DC	DC
EC	DC,EC PO,TPP^{-1} NTPP^{-1}	DC,EC PO,TPP TPP^{-1},EQ	DC,EC PO,TPP NTPP	EC,PO TPP NTPP	PO TPP NTPP	DC,EC	DC	EC
PO	DC,EC PO,TPP^{-1} NTPP^{-1}	DC,EC PO,TPP^{-1} NTPP^{-1}	*	PO TPP NTPP	PO TPP NTPP	DC,EC PO,TPP^{-1} NTPP^{-1}	DC,EC PO,TPP^{-1} NTPP^{-1}	PO
TPP	DC	DC,EC	DC,EC PO,TPP NTPP	TPP NTPP	NTPP	DC,EC PO,TPP TPP^{-1},EQ	DC,EC PO,TPP^{-1} NTPP^{-1}	TPP
NTPP	DC	DC	DC,EC PO,TPP NTPP	NTPP	NTPP	DC,EC PO,TPP NTPP	*	NTPP
TPP^{-1}	DC,EC PO,TPP^{-1} NTPP^{-1}	EC,PO TPP^{-1} NTPP^{-1}	PO TPP^{-1} NTPP^{-1}	PO,EQ TPP TPP^{-1}	PO TPP NTPP	TPP^{-1} NTPP^{-1}	NTPP^{-1}	TPP^{-1}
NTPP^{-1}	DC,EC PO,TPP^{-1} NTPP^{-1}	PO TPP^{-1} NTPP^{-1}	PO TPP^{-1} NTPP^{-1}	PO TPP^{-1} NTPP^{-1}	PO,TPP^{-1} TPP,NTPP NTPP^{-1},EQ	NTPP^{-1}	NTPP^{-1}	NTPP^{-1}
EQ	DC	EC	PO	TPP	NTPP	TPP^{-1}	NTPP^{-1}	EQ

Table 1: Composition table for the eight base relations of RCC-8, where $*$ specifies the universal relation.

whether it is possible to assign regions to the spatial variables in a way that all relations hold. We call this problem RSAT. When only relations of a specific set $\mathcal{S}$ are used in Θ, the corresponding reasoning problem is denoted by RSAT($\mathcal{S}$). RSAT is a Constraint Satisfaction Problem (CSP) [14], where variables are nodes and relations are arcs of the constraint graph. So RSAT can be solved using the standard methods developed for CSP's.

One approximative method for deciding consistency of a CSP is deciding its path-consistency [13]. A CSP is *path-consistent* iff for any consistent instantiation of two variables there exists a consistent instantiation of any third variable. Path-consistency is necessary but not sufficient for the consistency of a CSP, i.e. a CSP that is not path-consistent is not consistent, but a path-consistent CSP is not necessarily consistent. A CSP can be made path-consistent by successively removing relations from all edges with the following operation:

$$\forall k : R_{ij} \leftarrow R_{ij} \cap (R_{ik} \circ R_{kj})$$

where i, j, k are nodes and R_{ij} is the relation between i and j. If the empty relation occurs while performing this operation the CSP is not path-consistent, otherwise it is. Path-consistency can be decided in times $O(n^3)$ where n is the total number of nodes in the graph [15].

Other computational problems include RMIN, the problem of finding the minimal relations, i.e. the strongest implied relation for each pair of spatial regions, and RENT, the problem whether a spatial formula is entailed by Θ. These problems can be polynomially reduced to RSAT [10].

3 Encoding of RCC-8 in Modal Logic

Another way of solving problems concerning RCC is using the encoding of the relations in first order predicate logic. Such an encoding does not lead to efficient decision procedures, however. In order to overcome this problem, Bennett [2, 3] used different encodings of RCC-8 in propositional intuitionistic and modal logic. In this work we will use Bennett's encoding of RCC-8 in propositional modal logic [3]. An introduction to modal logic is given in Appendix A.

Bennett obtained this encoding by analyzing the relationship of regions to the universe $\mathcal{U}$. He restricted his analysis to closed regions that are connected if they share a point and overlap if they share an interior point. If, e.g, X and Y are disconnected, the complement of the intersection of X and Y is equal to the universe. Further, both regions must not be empty, i.e. the complements of both X and Y are not equal to the universe. In this way the eight base relations can be represented by constraints of the form $(m = \mathcal{U})$, called *model constraints*, and $(m \neq \mathcal{U})$, called *entailment constraints*, where m is a set-theoretic expression. Any model constraint must hold, whereas no entailment constraint must hold [2].

For some of the constraints it is necessary to refer to the interior of regions, therefore a topological interior operator i is needed, which must satisfy the following axioms [3]:

1. $i(X) \subseteq X$,
2. $i(i(X)) = i(X)$,
3. $i(\mathcal{U}) = \mathcal{U}$,
4. $i(X \cap Y) = i(X) \cap i(Y)$.

The model and entailment constraints can be encoded in modal logic, where regions correspond to propositional atoms and the interior operator i corresponds to a modal logical operator $\mathbf{I}$ (see Table 2). The axioms for i must also hold for the modal operator $\mathbf{I}$, which results in the following axioms [3] in modal logic:

1. $\mathbf{I}X \rightarrow X$,
2. $\mathbf{II}X \leftrightarrow \mathbf{I}X$,
3. $\mathbf{I}\top \leftrightarrow \top$ (for any tautology $\top$),
4. $\mathbf{I}(X \wedge Y) \leftrightarrow \mathbf{I}X \wedge \mathbf{I}Y$.

Relation	*Model Constraints*	*Entailment Constraints*
$\mathsf{DC}(X,Y)$	$\neg(X \wedge Y)$	$\neg X, \neg Y$
$\mathsf{EC}(X,Y)$	$\neg(\mathbf{I}X \wedge \mathbf{I}Y)$	$\neg(X \wedge Y), \neg X, \neg Y$
$\mathsf{PO}(X,Y)$	—	$\neg(\mathbf{I}X \wedge \mathbf{I}Y), X \rightarrow Y, Y \rightarrow X, \neg X, \neg Y$
$\mathsf{TPP}(X,Y)$	$X \rightarrow Y$	$X \rightarrow \mathbf{I}Y, Y \rightarrow X, \neg X, \neg Y$
$\mathsf{TPP}^{-1}(X,Y)$	$Y \rightarrow X$	$Y \rightarrow \mathbf{I}X, X \rightarrow Y, \neg X, \neg Y$
$\mathsf{NTPP}(X,Y)$	$X \rightarrow \mathbf{I}Y$	$Y \rightarrow X, \neg X, \neg Y$
$\mathsf{NTPP}^{-1}(X,Y)$	$Y \rightarrow \mathbf{I}X$	$X \rightarrow Y, \neg X, \neg Y$
$\mathsf{EQ}(X,Y)$	$X \rightarrow Y, Y \rightarrow X$	$\neg X, \neg Y$

Table 2: Modal logical encoding of the eight base relations [3].

Axioms 1 and 2 correspond to the modal logics T and $\mathsf{4}$ and axioms 3 and 4 already hold for any modal logic K, so $\mathbf{I}$ is a modal $\mathsf{S4}$-operator.

The four axioms specified by Bennett are not sufficient to exclude non-closed regions. In order to account for that, we add two formulas for each atom X, which correspond to topological properties of closed regions. A closed region is the closure of an open region and the complement of a closed region is an open region. $\neg X$ specifies the complement of X and $\neg\mathbf{I}\neg X$ the closure of X.

$$X \leftrightarrow \neg\mathbf{I}\neg\mathbf{I}X \tag{1}$$

$$\neg X \leftrightarrow \mathbf{I}\neg X \tag{2}$$

To combine the different model and entailment constraints, Bennett [3] uses another modal operator $\Box$. $\Box m$ is interpreted as $m = \mathcal{U}$ and $\neg\Box m$ as $m \neq \mathcal{U}$. Any model constraint m can be written as $\Box m$ and any entailment constraint as $\neg\Box m$. If $\Box X$ is true in a world w of a model $\mathcal{M}$, written as $(\mathcal{M}, w \Vdash \Box X)$, then X must be true in any world of $\mathcal{M}$. So $\Box$ is an $\mathsf{S5}$-operator with the constraint that all worlds are mutually accessible. Therefore Bennett calls it a strong $\mathsf{S5}$-operator [3]. So the encoding of $\mathsf{RCC\text{-}8}$ is done in multi-modal logic with an $\mathsf{S4}$-operator and a strong $\mathsf{S5}$-operator.

4 Computational Properties of RCC-8

In this section we prove that reasoning with RCC-8 as well as RCC-5 is NP-hard. Other complexity results on qualitative spatial reasoning were obtained by Grigni et al. [11] who worked with Egenhofer's topological relations [6]. These relations are defined using the so-called *9-intersection*, the 9 possible intersections of the interior, the boundary, and the complement of two regions. Apart from the more restricting definition of regions, 8 different relations can be defined which use the same distinctions of regions as RCC-8. Grigni et al [11]

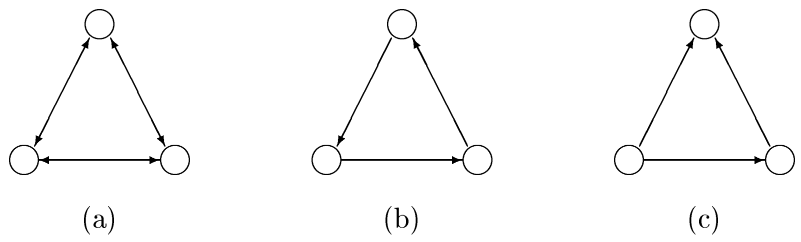

Figure 2: Property 4.1: (a) is the original configuration, (b) is impossible, only (c) is possible.

considered two different notions of satisfiability, the purely syntactical notion of *relational consistency* and the semantical notion of *realizability*, which are both different from what we call consistency. Relational consistency means that there is a path-consistent refinement of all relations to base relations, realizability means that there is a model consisting of simply connected planar regions. Both kinds of satisfiability were found to be NP-hard for Egenhofer's eight relations as well as a subset of five relations which uses different distinctions of regions than RCC-5.

We will prove NP-hardness for RCC-5 since the result can be transfered to RCC-8. The proof uses a transformation from NOT-ALL-EQUAL-3SAT to RSAT(RCC-5). NOT-ALL-EQUAL-3SAT is the problem of finding out whether there is a solution of the 3SAT problem, where every clause has at least one true and one false literal [9]. The transformation is based on two properties that can be verified using the composition table (Table 1). In the figures of this section $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$ is symbolized as $\longleftrightarrow$, $\{\mathsf{PP}\}$ as $\longrightarrow$, $\{\mathsf{PP}^{-1}\}$ as $\longleftarrow$ and $\{\mathsf{PO}\}$ as a dotted line.

Property 4.1 *Let X, Y, Z be regions, where the symmetrical relation $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$ holds between all of them. Then any refinement of these relations to base relations such that a path starting and ending at X passing Y and Z contains only $\{\mathsf{PP}\}$ or only $\{\mathsf{PP}^{-1}\}$ is inconsistent (see Figure 2).*

Property 4.2 *Let X, Y, Z be regions, where $X\{\mathsf{PP}, \mathsf{PP}^{-1}\}Z$, $Y\{\mathsf{PP}, \mathsf{PP}^{-1}\}Z$ and $X\{\mathsf{PO}\}Y$ hold. Then a refinement of these relations to base relations is only consistent, if between (Z, X) and (Z, Y) the same base relation holds (see Figure 3).*

Theorem 4.3 RSAT(RCC-5) *is* NP*-hard.*

Proof: Transformation of NOT-ALL-EQUAL-3SAT to RSAT(RCC-5) (see also Grigni et al [11]). Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be a set of variables and $\mathcal{C} =$

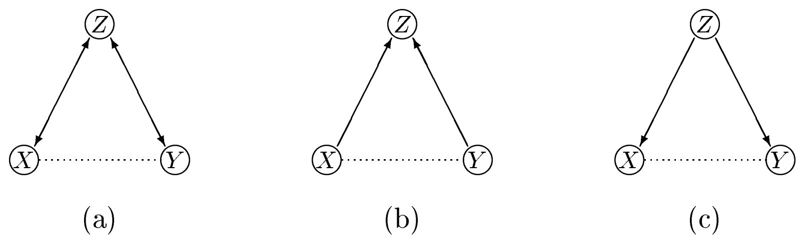

Figure 3: Property 4.2: (a) is the original configuration, (b) and (c) are the only possible refinements to base relations.

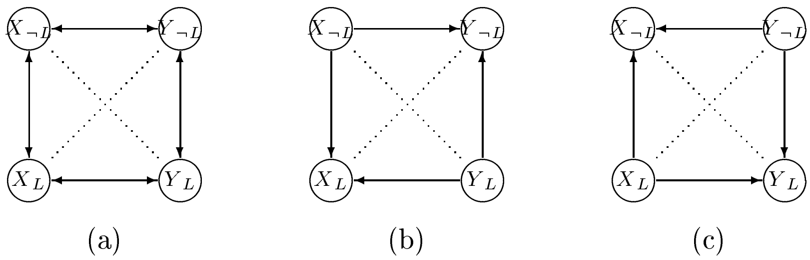

Figure 4: Transformation of variables and complementary literals, (b) and (c) are the only possible refinements to base relations.

$\{c_1, c_2, \ldots, c_m\}$ be a set of clauses of an arbitrary instance of NOT-ALL-EQUAL-3SAT with $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$, where $l_{i,j}$ are literal occurrences over variables of $\mathcal{V}$. We will construct a set of spatial formulas Θ, such that Θ is satisfiable iff $\mathcal{C}$ is a positive instance of NOT-ALL-EQUAL-3SAT using the following three transformation steps:

1. For each variable $v_L \in \mathcal{V}$ the regions $X_L, Y_L, X_{\neg L}$ and $Y_{\neg L}$ will be introduced by adding the following spatial formulas to Θ (see Figure 4):

$$X_L\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_L, X_{\neg L}\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_{\neg L},$$
$$X_L\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{\neg L}, Y_L\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_{\neg L},$$
$$X_L\{\mathsf{PO}\}Y_{\neg L}, Y_L\{\mathsf{PO}\}X_{\neg L}.$$

2. For each literal occurrence $l_{i,j}$ the regions $X_{i,j}$ and $Y_{i,j}$ will be introduced by adding the spatial formula $X_{i,j}\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_{i,j}$ to Θ. Depending on whether the literal occurrence is positive or negative, different spatial formulas have to be added to Θ.

   (a) $l_{i,j} \equiv v_L$:

$$X_{i,j}\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{\neg L}, Y_{i,j}\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_{\neg L},$$
$$X_{i,j}\{\mathsf{PO}\}Y_{\neg L}, Y_{i,j}\{\mathsf{PO}\}X_{\neg L}.$$

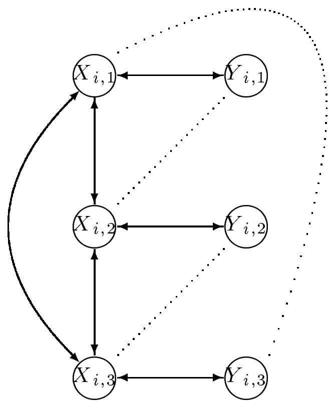

Figure 5: Transformation of a clause $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$ to spatial formulas.

(b) $l_{i,j} \equiv \neg v_L$:

$$X_{i,j}\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_L, Y_{i,j}\{\mathsf{PP}, \mathsf{PP}^{-1}\}Y_L,$$
$$X_{i,j}\{\mathsf{PO}\}Y_L, Y_{i,j}\{\mathsf{PO}\}X_L.$$

3. For each clause $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$ the following spatial formulas will be added to Θ:

$$X_{i,1}\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{i,2}, X_{i,2}\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{i,3}, X_{i,3}\{\mathsf{PP}, \mathsf{PP}^{-1}\}X_{i,1},$$
$$X_{i,1}\{\mathsf{PO}\}Y_{i,3}, X_{i,2}\{\mathsf{PO}\}Y_{i,1}, X_{i,2}\{\mathsf{PO}\}Y_{i,3}.$$

With this transformation for every variable as well as every literal occurrence two regions X and Y (with the appropriate indices) are introduced. When a literal occurrence or a variable is true, the corresponding regions hold the relation $X\{\mathsf{PP}\}Y$, when a literal occurrence or a variable is false, the corresponding regions hold the relation $X\{\mathsf{PP}^{-1}\}Y$.

Transformation step 1 introduces the regions corresponding to the positive and the negative literal of each variable. Property 4.2 assures that positive and negative literals have opposite assignments. This is shown in Figure 4. Transformation step 2 introduces regions for every literal occurrence. Again, Property 4.2 assures correct assignments. Finally, transformation step 3 together with Property 4.1 makes sure that the not-all-equal condition of the literals of every clause is also fulfilled by the corresponding regions. This is shown in Figure 5.

RSAT$\Rightarrow$ NOT-ALL-EQUAL-3SAT: Suppose that the CSP A obtained by transformation from a given instance B of NOT-ALL-EQUAL-3SAT is consistent, and suppose that α is a consistent instantiation of A. Then an assignment β that satisfies B can easily be obtained in the following way: For every variable $v_L \in \mathcal{V}$, if $X_L\{\mathsf{PP}\}Y_L$ holds, then $\beta(v_L)$ is true, otherwise $\beta(v_L)$ is false.

NOT-ALL-EQUAL-3SAT $\Rightarrow$ RSAT: Suppose that B is a positive instance of NOT-ALL-EQUAL-3SAT, and suppose that β is an assignment that satisfies B. Then the CSP A obtained by transformation from B is consistent. We will show this by constructing a spatial configuration that holds all relations of A. Before that we will point out some properties of A. Let A' be the CSP transformed from B with transformation steps 1 and 2. A' has the following properties:

1. Since transformation step 3 introduces no region, A' has the same regions as A.
2. For any variable $v_L \in \mathcal{V}$ the four corresponding regions are in a form as shown in Figure 4(b) or 4(c).
3. For any literal occurrence $l_{i,j}$ the two corresponding regions together with two regions of the affiliated variable are in a form as shown in Figure 4(b) or 4(c).
4. Only the relations $\{\mathsf{PP}\}, \{\mathsf{PP}^{-1}\}, \{\mathsf{PO}\}$ and $\{*\}$ occur in A'
5. If $X\{\mathsf{PP}\}Y$ holds for a region X and a region Y, then there is no region Z with $Z\{\mathsf{PP}\}X$.
6. If $X\{\mathsf{PP}^{-1}\}Y$ holds for a region X and a region Y, then there is no region Z with $Z\{\mathsf{PP}^{-1}\}X$.

Because of 5 and 6, all regions can be divided into two sets of regions, one with regions that are proper part of other regions, which we will call “small” regions, and one with regions that are converse proper part of other regions, which we will call “big” regions. The relation PO only holds between regions of the same set. All other relations are unspecified. For proving that A' is consistent, we will give a spatial configuration M' that holds all the specified relations and therefore is a model for A':

- Every small region S_i' consists of two disconnected parts s' and s_i', where s' is common to all small regions. $s_i'\{\mathsf{DR}\}s_j'$ for all $i \neq j$ and $s_i'\{\mathsf{DR}\}s'$ for all i.
- Every big region B_i' consists of two disconnected parts b' and b_i', where b' is common to all big regions. $b_i'\{\mathsf{DR}\}b_j'$ for all $i \neq j$ and $b_i'\{\mathsf{DR}\}b'$ for all i.
- For all i: $s_i'\{\mathsf{PP}\}b'$ and $s'\{\mathsf{PP}\}b'$.

A results from A' after applying transformation step 3. The spatial configuration M' is no model for A, because two relations of every clause do not hold. This can be seen in Figure 6 which shows the two possible assignments of a clause. Small regions are denoted as S_i, big regions as B_i. In every clause there is either one small region which is proper part of another small region or a big region which is proper part of another big region. In Figure 6 this is the case

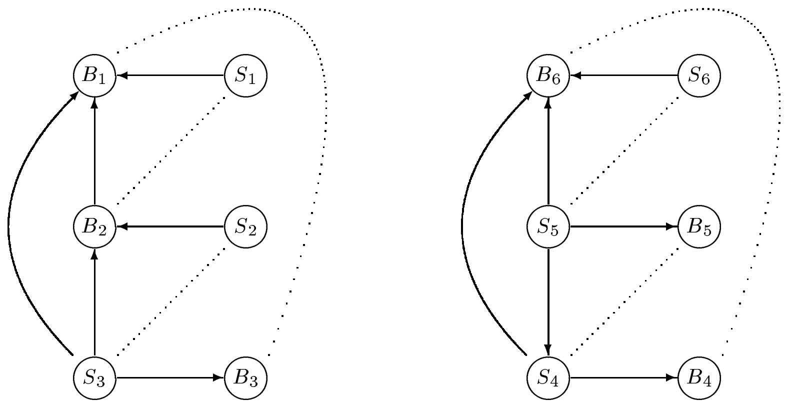

Figure 6: Two different types of not-all-equal clauses.

for $B_2\{\mathsf{PP}\}B_1$ and $S_4\{\mathsf{PP}^{-1}\}S_5$. Also there is one small region in every clause that partially overlaps one big region. In Figure 6 this is the case for $B_2\{\mathsf{PO}\}S_1$ and $S_4\{\mathsf{PO}\}B_5$.

With a few changes to M', we can construct a spatial configuration M that is a model for A. Apart from the following exceptions M is equal to M':

- For any i, j: If $S_i\{\mathsf{PP}^{-1}\}S_j$ then $S_i = s' \cup s'_i \cup s'_j$.
- For any i, j: If $B_i\{\mathsf{PP}^{-1}\}B_j$ then $B_i = b' \cup b'_i \cup b'_j$.
- For any i, j: If $B_i\{\mathsf{PO}\}S_j$ then $B_i = B'_i \setminus s'_j$.

As exactly two exceptions occur in every clause and the literal occurrences of every clause correspond to different regions, no region will be changed more than once. M holds all relations of A.

The transformation takes linear time in the number of clauses, so $\mathsf{RSAT}(\Theta)$ is NP-hard. $\square$

Since RCC-5 is a subset of RCC-8, this result can be easily applied to RCC-8. When $\{\mathsf{PP}\}$ is replaced by $\{\mathsf{TPP}, \mathsf{NTPP}\}$ and the same for their converse, properties 4.1 and 4.2 hold accordingly, and the same proof can be carried out.

Corollary 4.4 RSAT(RCC-8) *is* NP-*hard.*

A further complexity result on RCC-8 is from Nebel [16], who proved that RSAT can be decided efficiently if only base relations are used.

Theorem 4.5 (Nebel 1995) *Let $\mathcal{B}$ be the set of* RCC-8 *base relations. Then* $\mathsf{RSAT}(\mathcal{B}) \in \mathsf{P}$.

The proof uses Bennett's encoding of RCC-8 in propositional intuitionistic logic [2] for the transformation of $\mathsf{RSAT}(\mathcal{B})$ to 2-SAT.

In order to identify the borderline between tractability and intractability, one has to examine all subsets of RCC-8. We limit ourselves to subsets containing all base relations, because these subsets still allow to express definite knowledge, if it is available. Additionally, we require the universal relation to be in the subset, so that it is possible to express complete ignorance. This reduces the search space from 2^{256} subsets to 2^{247} subsets. Two other properties can be used to reduce the search space considerably. The first and obvious property states that subsets of tractable subsets are also tractable. The second property has likewise been used in finding the maximal tractable subset of Allen's calculus [18]. Since it can be applied to any set of relations, we will prove it in general. For this we have to define another problem, which we will call $\mathsf{CSPSAT}(\mathcal{S})$:

Instance: Subset $\mathcal{S}$ of a finite set $\mathcal{C}$ of binary relations over an infinite domain $\mathcal{D}$, where $\mathcal{C}$ is closed under composition, intersection, and converse and $\mathcal{S}$ contains the universal relation $\mathcal{D} \times \mathcal{D}$. Finite set $\mathcal{V}$ of variables over $\mathcal{D}$. Finite set Θ of formulas (XRY), where $R \in \mathcal{S}$ and $X, Y \in \mathcal{V}$.

Question: Is there an instantiation of all variables in Θ such that all formulas hold.

In the following $\widehat{\mathcal{S}}$ denotes the closure of $\mathcal{S}$ under composition, intersection and converse.

Theorem 4.6 $\mathsf{CSPSAT}(\widehat{\mathcal{S}})$ *can be polynomially reduced to* $\mathsf{CSPSAT}(\mathcal{S})$

Proof: Let $\mathcal{T} = \widehat{\mathcal{S}} \setminus \mathcal{S}$. Every element $R \in \mathcal{T}$ can be expressed by successive application of composition, intersection and converse of elements of $\mathcal{S}$. Let n be the maximal number of operations needed for a single element, where for each element the minimal number of operations is used.

We will show by induction that for any set Θ of formulas over $\widehat{\mathcal{S}}$ we can construct a set Θ' of formulas over $\mathcal{S}$ with $|\Theta'| \leq 2^n \times |\Theta|$ and Θ consistent iff Θ' consistent (induction hypothesis). Since n is fixed, the transformation is polynomial.

Base step ($n = 1$): Θ' contains all formulas $(X\,S\,Y) \in \Theta$ with $S \in \mathcal{S}$. For any formula $(X\,R\,Y) \in \Theta$ with $R \in \mathcal{T}$ one of the following cases applies:

1. $R = S^{\smile}$ and $S \in \mathcal{S}$. Add $(Y\,S\,X)$ to Θ'.
2. $R = S \circ T$ and $S, T \in \mathcal{S}$. Add $(X\,S\,Z)$ and $(Z\,T\,Y)$ to Θ', where Z is a new region.
3. $R = S \cap T$ and $S, T \in \mathcal{S}$. Add $(X\,S\,Y)$ and $(X\,T\,Y)$ to Θ'.

Then Θ' is consistent iff Θ is consistent, and $|\Theta'| \leq 2^1 \times |\Theta|$.

Inductive step: Suppose that $n = k+1$ and that the induction hypothesis holds for $n = k$. Let $\mathcal{T}' \subseteq \mathcal{T}$ be the set of relations that can only be composed of relations of $\mathcal{S}$ using $k+1$ operations. Then $\widehat{\mathcal{S}} \setminus \mathcal{T}'$ is the set of relations that can be composed of $\mathcal{S}$ using a maximal number of k operations. So the induction hypothesis is valid for $\widehat{\mathcal{S}} \setminus \mathcal{T}'$. Any relation $R \in \mathcal{T}'$ can be composed of relations of $\widehat{\mathcal{S}} \setminus \mathcal{T}'$ using exactly one operation, so R can be treated as specified in the base step. □

Since RSAT is a special case of CSPSAT, Theorem 4.6 can also be applied to RSAT.

Corollary 4.7 *Let $\mathcal{S}$ be a subset of* RCC-8*.*

1. $\mathsf{RSAT}(\widehat{\mathcal{S}}) \in \mathsf{P}$ *iff* $\mathsf{RSAT}(\mathcal{S}) \in \mathsf{P}$.
2. $\mathsf{RSAT}(\mathcal{S})$ *is* NP*-hard iff* $\mathsf{RSAT}(\widehat{\mathcal{S}})$ *is* NP*-hard.*

The first statement of Corollary 4.7 can be used to increase the number of elements of tractable subsets of RCC-8 considerably. With the second statement of Corollary 4.7 NP-hardness proofs of RSAT can be used to exclude certain relations from being in any tractable subset of RCC-8. The NP-hardness proof of Theorem 4.3, for example, only contains the relations $\{\mathsf{PO}\}$ and $\{\mathsf{TPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}, \mathsf{NTPP}^{-1}\}$ when written in RCC-8 relations. So for any subset $\mathcal{S}$ with the two relations contained in its closure $\widehat{\mathcal{S}}$, $\mathsf{RSAT}(\mathcal{S})$ is NP-hard.

A further NP-hardness proof of RSAT(RCC-8) can be specified to exclude more relations. This proof uses a polynomial transformation from ONE-IN-THREE-3SAT, the problem of finding a solution of 3SAT where exactly one literal is true in any clause [9]. Three more properties are necessary, which can be verified using the composition table (Table 1).

Property 4.8 *Let X, Y, Z be regions, where $X\{\mathsf{TPP}^{-1}, \mathsf{NTPP}\}Y$, $Y\{\mathsf{TPP}^{-1}, \mathsf{NTPP}\}Z$ and $Z\{\mathsf{TPP}^{-1}, \mathsf{NTPP}\}X$ hold. A refinement of these relations to base relations is only consistent, when either one or two of the relations are refined to $\{\mathsf{TPP}^{-1}\}$.*

Property 4.9 *Let X, Y, Z be regions, where $X\{\mathsf{TPP}^{-1}, \mathsf{NTPP}\}Z$, $Y\{\mathsf{TPP}^{-1}, \mathsf{NTPP}\}Z$ and $X\{\mathsf{PO}\}Y$ hold. A refinement of these relations to base relations is only consistent, when the relations between (X, Z) and between (Y, Z) are refined to the same base relation.*

Property 4.10 *Let X, Y, Z be regions, where $X\{\mathsf{TPP}^{-1}, \mathsf{NTPP}\}Z$, $Y\{\mathsf{TPP}, \mathsf{NTPP}^{-1}\}Z$ and $X\{\mathsf{PO}\}Y$ hold. A refinement of these relations to base relations is only consistent, when the relations between (X, Z) and between (Y, Z) are refined to different base relation.*

Lemma 4.11 *Let $\mathcal{S}$ be a subset of* RCC-8 *containing all base relations. If any of the relations $\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$, $\{\mathsf{TPP}, \mathsf{TPP}^{-1}\}$, $\{\mathsf{NTPP}, \mathsf{NTPP}^{-1}\}$, $\{\mathsf{NTPP}, \mathsf{TPP}^{-1}\}$ or $\{\mathsf{TPP}, \mathsf{NTPP}^{-1}\}$ is contained in $\widehat{\mathcal{S}}$, then $\mathsf{RSAT}(\mathcal{S})$ is* NP*-hard.*

Proof: The proof for $\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$ can be obtained by replacing the RCC-5 relations of Theorem 4.3 with the corresponding RCC-8 relations. The relations $\{\mathsf{TPP}, \mathsf{TPP}^{-1}\}$ and $\{\mathsf{NTPP}, \mathsf{NTPP}^{-1}\}$ also hold Properties 4.1 and 4.2, so the transformation of Theorem 4.3 can also be applied using these relations instead of $\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$.
Using the relation $\{\mathsf{NTPP}, \mathsf{TPP}^{-1}\}$ and Properties 4.8, 4.9, and 4.10, RSAT can be polynomially transformed to ONE-IN-THREE-3SAT with almost the same transformation steps as specified in Theorem 4.3. Within these steps $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$ has to be replaced with $\{\mathsf{NTPP}, \mathsf{TPP}^{-1}\}$, $\{\mathsf{PP}\}$ with $\{\mathsf{NTPP}\}$ and $\{\mathsf{PP}^{-1}\}$ with $\{\mathsf{TPP}^{-1}\}$. Because of Property 4.8 exactly one literal must be true in any clause. If $\{\mathsf{TPP}, \mathsf{NTPP}^{-1}\}$ is contained in $\widehat{\mathcal{S}}$, then $\{\mathsf{TPP}^{-1}, \mathsf{NTPP}\}$ is also contained. With Corollary 4.7 the proof is completed. □

By computing the closure of all sets containing the eight base relations together with one additional relation, the following lemma can be obtained.

Lemma 4.12 RSAT($\mathcal{S}$) *is* NP-*hard for any subset* $\mathcal{S}$ *of* RCC-8 *containing all base relations together with one of the* 72 *relations of the following sets:*

$$\begin{aligned}
\mathcal{N}_1 &= \{R \,|\, \{\mathsf{PO}\} \not\subseteq R \textit{ and } (\{\mathsf{TPP}, \mathsf{TPP}^{-1}\} \subseteq R \textit{ or } \{\mathsf{NTPP}, \mathsf{NTPP}^{-1}\} \subseteq R)\},\\
\mathcal{N}_2 &= \{R \,|\, \{\mathsf{PO}\} \not\subseteq R \textit{ and } (\{\mathsf{TPP}, \mathsf{NTPP}^{-1}\} \subseteq R \textit{ or } \{\mathsf{TPP}^{-1}, \mathsf{NTPP}\} \subseteq R)\}.
\end{aligned}$$

Proof: The closure of any of the 72 subsets contains one of the five relations of Lemma 4.11. □

5 Transformation of RSAT to SAT

For transforming RSAT to propositional satisfiability (SAT) we will transform every instance Θ of RSAT to a propositional formula in conjunctive normal form (CNF) that is satisfiable iff Θ is consistent. We will start from $m(\Theta)$, the multi-modal encoding of Θ, and show that whenever $m(\Theta)$ is satisfiable, it has a Kripke model of a specific type. This model will then be used to transform $m(\Theta)$ to a propositional formula.

Let in the following Θ be a set of RCC-8 formulas and $Reg(\Theta)$ be the set of spatial variables used in Θ, then $m(\Theta)$ specifies the multi-modal encoding of Θ, where

$$m(\Theta) = \left(\bigwedge_{XRY\in\Theta} m_1(XRY)\right) \wedge \left(\bigwedge_{X\in Reg(\Theta)} m_2(X)\right).$$

The multi-modal encoding m_1 of a spatial formula XRY is determined by the base relations B contained in R:

$$m_1(XRY) = \bigvee_{B\subseteq R} m_1(XBY).$$

The multi-modal encoding of spatial formulas containing only base relations results directly from Table 2:

$$\begin{aligned}
m_1(X\{\mathsf{DC}\}Y) &= \Box(\neg(X \wedge Y)),\\
m_1(X\{\mathsf{EC}\}Y) &= \Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y)) \wedge \neg\Box(\neg(X \wedge Y)),\\
m_1(X\{\mathsf{PO}\}Y) &= \neg\Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y)) \wedge \neg\Box(X \rightarrow Y) \wedge \neg\Box(Y \rightarrow X),\\
m_1(X\{\mathsf{TPP}\}Y) &= \Box(X \rightarrow Y) \wedge \neg\Box(X \rightarrow \mathbf{I}Y) \wedge \neg\Box(Y \rightarrow X),\\
m_1(X\{\mathsf{TPP}^{-1}\}Y) &= \Box(Y \rightarrow X) \wedge \neg\Box(Y \rightarrow \mathbf{I}X) \wedge \neg\Box(X \rightarrow Y),\\
m_1(X\{\mathsf{NTPP}\}Y) &= \Box(X \rightarrow \mathbf{I}Y) \wedge \neg\Box(Y \rightarrow X),\\
m_1(X\{\mathsf{NTPP}^{-1}\}Y) &= \Box(Y \rightarrow \mathbf{I}X) \wedge \neg\Box(X \rightarrow Y),\\
m_1(X\{\mathsf{EQ}\}Y) &= \Box(X \rightarrow Y) \wedge \Box(Y \rightarrow X).
\end{aligned}$$

m_2 consists of the modal formulas that have to be true for every region $X \in Reg(\Theta)$ and results from the axioms of the **I**-operator, the additional properties of closed regions and the entailment constraint $\neg X$.

$$\begin{aligned}
m_2(X) &= \Box(\mathbf{I}X \rightarrow X) \wedge \Box(\mathbf{I}X \rightarrow \mathbf{II}X)\\
&\quad \wedge\Box(\neg X \rightarrow \mathbf{I}\neg X) \wedge \Box(X \rightarrow \neg\mathbf{I}\neg\mathbf{I}X) \wedge \neg\Box\neg X.
\end{aligned}$$

Note that the formulas $\Box(\mathbf{II}X \rightarrow \mathbf{I}X)$, $\Box(\mathbf{I}\neg X \rightarrow \neg X)$ and $\Box(\neg\mathbf{I}\neg\mathbf{I}X \rightarrow X)$ are entailed by the other formulas of m_2 and can therefore be ignored. As follows from the work by Bennett [3], Θ is consistent iff $m(\Theta)$ is satisfiable.

$m(\Theta)$ is satisfiable if it is true in a world w of a Kripke model $\mathcal{M} = \langle W, \{R_1 = W \times W, R_2 \subseteq W \times W\}, \pi\rangle$, where W is a set of worlds, R_1 the accessibility relation of the $\Box$-operator, R_2 the accessibility relation of the **I**-operator, and π a truth function that assigns a truth value to every atom in every world. The truth conditions for $\mathcal{M}, w \Vdash m(\Theta)$ can be specified as a combination of truth conditions of sub-formulas according to the form of $m(\Theta)$.[2]

$$\begin{aligned}
\mathcal{M}, w \Vdash m(\Theta) \quad &\text{iff} \quad \mathcal{M}, w \Vdash \bigwedge_{XRY\in\Theta} m_1(XRY) \wedge\\
&\qquad\ \mathcal{M}, w \Vdash \bigwedge_{X\in Reg(\Theta)} m_2(X)\\
&\text{iff} \quad \mathcal{M}, w \Vdash \bigwedge_{XRY\in\Theta} \bigvee_{B\subseteq R} m_1(XBY) \wedge\\
&\qquad\ \mathcal{M}, w \Vdash \bigwedge_{X\in Reg(\Theta)} m_2(X)\\
&\text{iff} \quad \bigwedge_{XRY\in\Theta} \bigvee_{B\subseteq R} \mathcal{M}, w \Vdash m_1(XBY) \wedge\\
&\qquad\ \bigwedge_{X\in Reg(\Theta)} \mathcal{M}, w \Vdash m_2(X).
\end{aligned}$$

We can carry on expressing the truth conditions of $m(\Theta)$ by combining the truth conditions of sub-formulas until we have only combinations of formulas

[2] The logical operators used to combine expressions of the form $\mathcal{M}, w \Vdash \varphi$ are not the usual modal operators, but operators in a meta language.

of the type $\mathcal{M}, w \Vdash \Box\varphi$ or $\mathcal{M}, w \Vdash \neg\Box\varphi$. These sub-formulas result from the different modal and entailment constraints, the axioms for the $\mathbf{I}$ operator, and the properties of closed regions and form 18 structurally different formulas. The truth conditions of these formulas can be obtained by combining the truth values of the single atoms. Since $\Box$ is a strong S5-operator and every world is accessible by any other world, the condition *for all u with wR_1u* can be replaced with $\forall u$.

$$\begin{array}{rll}
\mathcal{M}, w \Vdash \Box(\neg(X \wedge Y)) & \text{iff} & \forall u.\,(\mathcal{M}, u \nVdash X \vee \mathcal{M}, u \nVdash Y) \quad (3)\\
\mathcal{M}, w \Vdash \Box(X \rightarrow Y) & \text{iff} & \forall u.\,(\mathcal{M}, u \nVdash X \vee \mathcal{M}, u \Vdash Y) \quad (4)\\
\mathcal{M}, w \Vdash \Box(Y \rightarrow X) & \text{iff} & \forall u.\,(\mathcal{M}, u \Vdash X \vee \mathcal{M}, u \nVdash Y) \quad (5)\\
\mathcal{M}, w \Vdash \neg\Box(\neg(X \wedge Y)) & \text{iff} & \exists u.\,(\mathcal{M}, u \Vdash X \wedge \mathcal{M}, u \Vdash Y) \quad (6)\\
\mathcal{M}, w \Vdash \neg\Box(X \rightarrow Y) & \text{iff} & \exists u.\,(\mathcal{M}, u \Vdash X \wedge \mathcal{M}, u \nVdash Y) \quad (7)\\
\mathcal{M}, w \Vdash \neg\Box(Y \rightarrow X) & \text{iff} & \exists u.\,(\mathcal{M}, u \nVdash X \wedge \mathcal{M}, u \Vdash Y) \quad (8)\\
\mathcal{M}, w \Vdash \neg\Box\neg X & \text{iff} & \exists u.\,(\mathcal{M}, u \Vdash X) \quad (9)\\
\mathcal{M}, w \Vdash \neg\Box\neg Y & \text{iff} & \exists u.\,(\mathcal{M}, u \Vdash Y) \quad (10)\\
\mathcal{M}, w \Vdash \Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y)) & \text{iff} & \forall u.(\mathcal{M}, u \nVdash \mathbf{I}X \vee \mathcal{M}, u \nVdash \mathbf{I}Y) \quad (11)\\
& \text{iff} & \forall u.\exists v : uR_2v.(\mathcal{M}, v \nVdash X \vee \mathcal{M}, v \nVdash Y)\\
\mathcal{M}, w \Vdash \Box(X \rightarrow \mathbf{I}Y) & \text{iff} & \forall u.(\mathcal{M}, u \nVdash X \vee \mathcal{M}, u \Vdash \mathbf{I}Y) \quad (12)\\
& \text{iff} & \forall u.\forall v : uR_2v.(\mathcal{M}, u \nVdash X \vee \mathcal{M}, v \Vdash Y)\\
\mathcal{M}, w \Vdash \Box(Y \rightarrow \mathbf{I}X) & \text{iff} & \forall u.(\mathcal{M}, u \Vdash \mathbf{I}X \vee \mathcal{M}, u \nVdash Y) \quad (13)\\
& \text{iff} & \forall u.\forall v : uR_2v.(\mathcal{M}, v \Vdash X \vee \mathcal{M}, u \nVdash Y)\\
\mathcal{M}, w \Vdash \neg\Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y)) & \text{iff} & \exists u.(\mathcal{M}, u \Vdash \mathbf{I}X \wedge \mathcal{M}, u \Vdash \mathbf{I}Y) \quad (14)\\
& \text{iff} & \exists u.\forall v : uR_2v.(\mathcal{M}, v \Vdash X \wedge \mathcal{M}, v \Vdash Y)\\
\mathcal{M}, w \Vdash \neg\Box(X \rightarrow \mathbf{I}Y) & \text{iff} & \exists u.(\mathcal{M}, u \Vdash X \wedge \mathcal{M}, u \nVdash \mathbf{I}Y) \quad (15)\\
& \text{iff} & \exists u.\exists v : uR_2v.(\mathcal{M}, u \Vdash X \wedge \mathcal{M}, v \nVdash Y)\\
\mathcal{M}, w \Vdash \neg\Box(Y \rightarrow \mathbf{I}X) & \text{iff} & \exists u.(\mathcal{M}, u \nVdash \mathbf{I}X \wedge \mathcal{M}, u \Vdash Y) \quad (16)\\
& \text{iff} & \exists u.\exists v : uR_2v.(\mathcal{M}, v \nVdash X \wedge \mathcal{M}, u \Vdash Y)\\
\mathcal{M}, w \Vdash \Box(\mathbf{I}X \rightarrow X) & \text{iff} & \forall u.(\mathcal{M}, u \nVdash \mathbf{I}X \vee \mathcal{M}, u \Vdash X) \quad (17)\\
& \text{iff} & \forall u.\exists v : uR_2v.(\mathcal{M}, v \nVdash X \vee \mathcal{M}, u \Vdash X)\\
\mathcal{M}, w \Vdash \Box(\mathbf{I}X \rightarrow \mathbf{II}X) & \text{iff} & \forall u.(\mathcal{M}, u \nVdash \mathbf{I}X \vee \mathcal{M}, u \Vdash \mathbf{II}X) \quad (18)\\
& \text{iff} & \forall u.((\exists v : uR_2v.\mathcal{M}, v \nVdash X)\\
& & \quad \vee (\forall s : uR_2s.\forall t : sR_2t.\mathcal{M}, t \Vdash X))\\
\mathcal{M}, w \Vdash \Box(\neg X \rightarrow \mathbf{I}\neg X) & \text{iff} & \forall u.(\mathcal{M}, u \Vdash X \vee \mathcal{M}, u \Vdash \mathbf{I}\neg X) \quad (19)\\
& \text{iff} & \forall u.\forall v : uR_2v.(\mathcal{M}, u \Vdash X \vee \mathcal{M}, v \nVdash X)\\
\mathcal{M}, w \Vdash \Box(X \rightarrow \neg\mathbf{I}\neg\mathbf{I}X) & \text{iff} & \forall u.(\mathcal{M}, u \nVdash X \vee \mathcal{M}, u \nVdash \mathbf{I}\neg\mathbf{I}X) \quad (20)\\
& \text{iff} & \forall u.\exists s : uR_2s.\forall t : sR_2t.\\
& & \quad (\mathcal{M}, u \nVdash X \vee \mathcal{M}, t \Vdash X)
\end{array}$$

We will call this form of writing $m(\Theta)$ the *explicit form* of $m(\Theta)$. In order to keep the further analysis more readable we will introduce some abbreviations for the model and entailment constraints.

Definition 5.1 *Abbreviations for the model constraints:*

$$\begin{array}{lcl|lcl} \alpha_{xy} & \equiv & \Box(\neg(X \wedge Y)) & A_{xy} & \equiv & \Box(\neg(\mathbf{I}X \wedge \mathbf{I}Y)) \\ \beta_{xy} & \equiv & \Box(X \rightarrow Y) & B_{xy} & \equiv & \Box(X \rightarrow \mathbf{I}Y) \\ \gamma_{xy} & \equiv & \Box(Y \rightarrow X) & C_{xy} & \equiv & \Box(Y \rightarrow \mathbf{I}X). \end{array}$$

As the entailment constraints are negations of the model constraints, they will be abbreviated as negations of the above abbreviations. When it is obvious which atoms are used, the abbreviations will be written without indices. The entailment constraints $\neg\Box X$ and $\neg\Box Y$ can be written as a combination of abbreviations, namely, $\neg\alpha\vee\neg\beta$ and $\neg\alpha\vee\neg\gamma$. Not all combinations of abbreviations are possible, as some of them result in a contradiction or force others.

Lemma 5.2 *The following modal formulas written as abbreviations are inconsistent, the negations of these formulas are tautologies:*

$$\alpha \wedge \beta, \alpha \wedge \gamma, \alpha \wedge \neg A, \neg\beta \wedge B, \neg\gamma \wedge C,$$
$$\beta \wedge A, \gamma \wedge A, \gamma \wedge B \wedge \neg C, \beta \wedge \neg B \wedge C.$$

Proof: Using e.g. tableau based proof procedures for multi-modal logic [7], a contradiction can be inferred from each of the formulas.

$\alpha \wedge \beta$: contradicts the fact that no region is empty.

$\alpha \wedge \gamma$: analogous.

$\alpha \wedge \neg A$: results in a contradiction when combined with $\mathbf{I}X \rightarrow X$.

$\neg\beta \wedge B$: results in a contradiction when combined with $\neg X \rightarrow \neg\mathbf{I}X$.

$\neg\gamma \wedge C$: analogous.

$\beta \wedge A$: Because of $(X \rightarrow \neg\mathbf{I}\neg\mathbf{I}X)$ and because no region is empty, there is a world u with $\mathcal{M}, u \Vdash \mathbf{I}X$. Because of $(\mathbf{I}X \rightarrow X)$, $\mathcal{M}, u \Vdash X$ also holds. β entails $\mathcal{M}, u \Vdash Y$ and A entails $\mathcal{M}, u \nVdash \mathbf{I}Y$. So there is an R_2-successor v of u with $\mathcal{M}, v \nVdash Y$. Because of $\mathcal{M}, u \Vdash \mathbf{I}X$, $\mathcal{M}, v \Vdash X$ also holds. β entails $\mathcal{M}, v \Vdash Y$, which results in a contradiction.

$\gamma \wedge A$: analogous.

$\gamma \wedge B \wedge \neg C$: Because of $\neg C$ there is a world u with $\mathcal{M}, u \nVdash \mathbf{I}X$ and $\mathcal{M}, u \Vdash Y$. Because of γ, $\mathcal{M}, u \Vdash X$ holds and because of B, $\mathcal{M}, u \Vdash \mathbf{I}Y$ holds. Since $\mathcal{M}, u \nVdash \mathbf{I}X$ holds, there is an R_2-successor v of u with $\mathcal{M}, v \nVdash X$. So γ results in $\mathcal{M}, v \nVdash Y$ and $\mathcal{M}, u \Vdash \mathbf{I}Y$ results in $\mathcal{M}, v \Vdash Y$, which is a contradiction.

$\beta \wedge \neg B \wedge C$: analogous. $\Box$

The formula $\beta \wedge C$ implies that $\mathbf{I}X \leftrightarrow X$ holds, which can only be true if X is a tautology, i.e. region X is the universe. The same holds for $\gamma \wedge B$ and region Y.

Now it is possible to write the modal encoding $m_1(XRY)$ of every relation R of RCC-8 using only abbreviations of Definition 5.1. We will call this encoding the *abbreviated form* of R. The abbreviated form of the eight base relations is the following:

$$\begin{aligned}
m_1(X\{\mathsf{DC}\}Y) &\equiv \alpha \\
m_1(X\{\mathsf{EC}\}Y) &\equiv \neg\alpha \wedge A \\
m_1(X\{\mathsf{PO}\}Y) &\equiv \neg\beta \wedge \neg\gamma \wedge \neg A \\
m_1(X\{\mathsf{TPP}\}Y) &\equiv \beta \wedge \neg\gamma \wedge \neg B \\
m_1(X\{\mathsf{TPP}^{-1}\}Y) &\equiv \neg\beta \wedge \gamma \wedge \neg C \\
m_1(X\{\mathsf{NTPP}\}Y) &\equiv \neg\gamma \wedge \neg B \\
m_1(X\{\mathsf{NTPP}^{-1}\}Y) &\equiv \neg\beta \wedge C \\
m_1(X\{\mathsf{EQ}\}Y) &\equiv \beta \wedge \gamma
\end{aligned}$$

The abbreviated form of the other relations can be obtained by conjunctively connecting the abbreviated form of the contained base relations. When the abbreviations are regarded as "propositional atoms", we can write the abbreviated form of all relations in conjunctive normal form.

It remains to be proven that a polynomial model can be found. We will construct a model with polynomial many worlds that satisfies $m(\Theta)$ if it is satisfiable at all. Therefore we will only introduce worlds that are explicitly needed. For this we will have to take a closer look at formulas (3) – (20). Amongst them are formulas of the type $\forall w.\varphi$ and formulas of the type $\exists w.\varphi$. Only formulas of the type $\exists w.\varphi$ explicitly require a world with the specified properties φ. If a world with these properties is not present, it has to be introduced. Formulas of the type $\forall w.\varphi$ restrict the properties of worlds already introduced. So worlds have to be introduced only for existential quantifiers. Since there are also formulas of the type $\forall w.\exists v.\phi$ a new world may not be introduced for every occurring existential quantifier. In the following we will analyze which existential quantifier introduces a new world and for which there already exists a world with the specified properties.

Definition 5.3 *Let $u \in W$ be a world of the model $\mathcal{M}$.*

- *u is a world of level 0 if vR_2u only holds for $v = u$.*
- *u is a world of level $l+1$ if vR_2u holds for a world v of level l and there is no world $v \neq u$ of level $> l$.*
- *If w is a world of level 0 and v is an R_2-successor of w, then w is called the introductory world of v.*
- *$W_i \subseteq W$ is the set of worlds of level i.*

Every occurrence of one of formulas (6) to (8) or (14) to (16), which correspond to the six negated abbreviations, shall introduce a new world of level 0. Formulas (15) and (16) introduce additional worlds of level 1. Formulas (17) and (18) do not need new worlds as every S4-frame already fulfills these conditions. Every model shall be designed according to these principles.

Definition 5.4 *Let Θ be a set of* RCC-8 *formulas and $n = |Reg(\Theta)|$ be the number of regions in Θ. Then an* RCC-8*-frame $\mathcal{F} = \langle W, \{R_1, R_2\}\rangle$ has the following properties:*

1. *There are only worlds of level $0, 1$ and 2.*
2. *For every world u of level k ($k = 0, 1$) there are exactly $2n$ worlds v of level $k + 1$ with uR_2v.*
3. *For every world v of level k there is exactly one world u for every level $0 \leq l \leq k$ with uR_2v.*

An RCC-8*-model is based on an* RCC-8*-frame. In a polynomial* RCC-8*-frame the number of worlds of level 0 is polynomially bounded by n.*

Lemma 5.5 *If $m(\Theta)$ is satisfiable, then there is a polynomial* RCC-8*-model $\mathcal{M}$ with $\mathcal{M}, w \Vdash m(\Theta)$.*

Proof: Every relation contains at most 6 different negated abbreviations,[3] which introduce a new world of level 0. As there are n regions and therefore n^2 relations, there are at most $6n^2$ worlds of level 0. For each world w and each atom X there might be a formula that forces the existence of an R_2-successor that forces X or forces $\neg X$. So a maximal number of $2n$ different R_2-successors is necessary for each world.

Suppose that $\mathcal{M}' = \langle W', R', \pi'\rangle$ models $m(\Theta)$ and there is no polynomial RCC-8-model that models $m(\Theta)$. We will prove by contradiction that a polynomial RCC-8-model exists if $m(\Theta)$ is satisfiable. For this we will construct a polynomial RCC-8-model $\mathcal{M} = \langle W, R, \pi\rangle$ using $\mathcal{M}'$. Let $W = W_0' \cup W_1' \cup W_2'$, $R = \{(u, v) \in R' | u, v \in W\}$ and $\pi(w, a) = \pi'(w, a)$ for all $w \in W$ and all propositional atoms a. We now have to find out which of the formulas (3) to (20) hold for $\mathcal{M}'$ but not for $\mathcal{M}$. Trivially, if formulas (3) to (10) hold for $\mathcal{M}'$ they also hold for $\mathcal{M}$. The same for (14) to (16). As the only R_2-successor of a world of level 2 is the world itself, formulas (12), (13) and (17) to (20) also hold for $\mathcal{M}$ if they hold for $\mathcal{M}'$.

Suppose that formula (11) holds for X and Y and there is a world $v \in W_2'$ with $\pi'(v, X) =$ true and $\pi'(v, Y) =$ true. In this case $\mathcal{M}$ is not a model for $m(\Theta)$,

[3] This number results from a straightforwardly computed abbreviated form of every relation, where all tautologies, as specified in Lemma 5.2, were eliminated. It might be decreased by optimizing the abbreviated form with respect to the number of negated abbreviations.

but it would be a model if the truth value of either $\pi(v, X)$ or $\pi(v, Y)$ is set to false and this does not contradict any other formula. Therefore it is necessary to find out which formula forces $\pi(w, X)$ to be true for a world w of level 2 and atom X. Formulas (4) and (5) force it only when another atom Z is already forced to be true in w. If X or Y is forced by one of the formulas (12) to (14), (18) or (20) to be true in w then it is also forced by all R_2-successors of w.
If both X and Y are forced to be true in v, they must both be true in any R_2-successor of v and $\mathcal{M}'$ can not be a model for $m(\Theta)$, so one of them, say X, is not forced and $\pi(v, X)$ can be set to false. We have constructed a polynomial RCC-8-model which contradicts our initial assumption. □

We proved that whenever $m(\Theta)$ is satisfiable, a model for $m(\Theta)$ can be based on a polynomial RCC-8-frame. We will now transform the explicit form of $m(\Theta)$ to a propositional formula $p(m(\Theta))$ in CNF such that $p(m(\Theta))$ is satisfiable iff $m(\Theta)$ is satisfiable in a polynomial RCC-8-frame $\mathcal{F} = \langle W, \{R_1, R_2\}\rangle$. For this we have to introduce a propositional atom for every world $w \in W$ and every region $X \in Reg(\Theta)$. To preserve the structure of the RCC-8-frame in propositional logic, two functions have to be defined:

Definition 5.6 *Let $\mathcal{F} = \langle W, \{R_1, R_2\}\rangle$ be an RCC-8-frame.*

- *$f : W \to W_0$ determines the introductory world of every world,*
- *$g : W \to \emptyset \cup \{1 \dots 2n\} \cup (\{1 \dots 2n\} \times \{1 \dots 2n\})$ provides all worlds of the same introductory world with a specific order, i.e. if the worlds u and v are distinct, have the same level and $f(u) = f(v)$, then $g(u) \neq g(v)$. If u has level 0 then $g(u) = \emptyset$, if u has level 1 then $g(u) \in \{1 \dots 2n\}$ and if u has level 2 then $g(u) \in \{1 \dots 2n\} \times \{1 \dots 2n\}$. If uR_2v and u is no introductory world, $g(u)$ is the first argument of $g(v)$.*

For every world $w \in W$ and every region $X \in Reg(\Theta)$ the propositional atom $X_{f(w)}^{g(w)}$ will be introduced.

Proposition 5.7 *The explicit form of $m(\Theta)$ can be transformed to a propositional formula $q(m(\Theta))$ as follows:*

- *$\mathcal{M}, w \Vdash X$ is transformed to $X_{f(w)}^{g(w)}$.*
- *$\mathcal{M}, w \not\Vdash X$ is transformed to $\neg X_{f(w)}^{g(w)}$.*
- *The meta operators $\wedge$ and $\vee$ (see Footnote 2) are transformed to the propositional operators $\wedge$ and $\vee$, respectively.*

Since all worlds of the frame are known, quantifiers can be transformed to propositional operators:

- *A universal quantification of particular worlds results in a conjunction over these worlds.*
- *An existential quantification of a world of level 0 results in a disjunction over all worlds of level 0.*
- *An existential quantification of an R_2–successor of world w results in a disjunction over $2n$ R_2–successor of w.*

It is easy to see that with this transformation $X^{g(w)}_{f(w)} = \text{true}$ iff $\pi(X, w) = \text{true}$.

With the specified transformation $q(\alpha), q(\neg\alpha)$ and $q(A)$, for example, can be written as:

$$
\begin{aligned}
q(\alpha) &= \bigwedge_{w \in W_0} (\neg X_w \vee \neg Y_w) \wedge \bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} (\neg X^i_w \vee \neg Y^i_w) \wedge \\
&\quad \bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} \bigwedge_{j=1}^{2n} (\neg X^{i,j}_w \vee \neg Y^{i,j}_w), \\
q(\neg\alpha) &= \bigvee_{w \in W_0} (X_w \wedge Y_w), \\
q(A) &= \bigwedge_{w \in W_0} \bigvee_{i=1}^{2n} (\neg X^i_w \vee \neg Y^i_w) \wedge \bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} \bigvee_{j=1}^{2n} (\neg X^{i,j}_w \vee \neg Y^{i,j}_w).
\end{aligned}
$$

In order to obtain a propositional formula in CNF, existential quantifiers must be treated differently. As stated before, the negated abbreviations introduce new worlds of level 0, so instead of a disjunction over all worlds of level 0, a new world of level 0 together with it's $4n^2$ R_2-successors will be explicitly introduced. $q(\neg\alpha)$, for example, can now be written as $X_w \wedge Y_w$ (*for a new world w of level 0*).

$2n$ R_2-successors are necessary such that for any statement of the form $\exists u.wR_2u : \mathcal{M}, u \Vdash X$ or $\exists u.wR_2u : \mathcal{M}, u \not\Vdash X$ there exists a world that makes this statement true. We will reserve one of the $2n$ R_2-successors for every statement of this kind. Therefore we need a function $h : Reg(\Theta) \cup \overline{Reg(\Theta)} \rightarrow \{1 \ldots 2n\}$ that associates every region and the complement of every region with one of the $2n$ R_2-successors. If there exists an R_2-successor u of w with $\mathcal{M}, u \Vdash X$, then $X^{h(x)}_w = \text{true}$. If there exists an R_2-successor u of w with $\mathcal{M}, u \not\Vdash X$, then $X^{h(\neg x)}_w = \text{true}$. If the $h(X)$-successor of a world w holds $\neg X$ then every R_2-successor of w holds $\neg X$. If the $h(\neg X)$-successor of w holds X then every R_2-successor of w holds X. To ensure these properties, additional formulas have to be specified for every region X:

$$
\bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} (\neg X^{h(\neg x)}_w \vee X^i_w) \wedge \bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} \bigwedge_{j=1}^{2n} (\neg X^{i,h(\neg x)}_w \vee X^{i,j}_w) \tag{21}
$$

$$
\bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} (X^{h(x)}_w \vee \neg X^i_w) \wedge \bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} \bigwedge_{j=1}^{2n} (X^{i,h(x)}_w \vee \neg X^{i,j}_w) \tag{22}
$$

With these modifications to $q(m(\Theta))$ we obtain a propositional formula $p(m(\Theta))$ in conjunctive normal form. The transformation is given in the following proposition.

Proposition 5.8 *All abbreviations, their negations, the axioms for the* $\mathbf{I}$*-operator and the properties of closed regions can be transformed to propositional logic in the following way:*

$$p(\alpha) = \bigwedge_{w\in W_0}(\neg X_w \vee \neg Y_w) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}(\neg X_w^i \vee \neg Y_w^i) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\bigwedge_{j=1}^{2n}(\neg X_w^{i,j} \vee \neg Y_w^{i,j})$$

$$p(\neg\alpha) = X_w \wedge Y_w \text{ (for a new world } w \text{ of level 0)}$$

$$p(\beta) = \bigwedge_{w\in W_0}(\neg X_w \vee Y_w) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}(\neg X_w^i \vee Y_w^i) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\bigwedge_{j=1}^{2n}(\neg X_w^{i,j} \vee Y_w^{i,j})$$

$$p(\neg\beta) = X_w \wedge \neg Y_w \text{ (for a new world } w \text{ of level 0)}$$

$$p(\gamma) = \bigwedge_{w\in W_0}(X_w \vee \neg Y_w) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}(X_w^i \vee \neg Y_w^i) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\bigwedge_{j=1}^{2n}(X_w^{i,j} \vee \neg Y_w^{i,j})$$

$$p(\neg\gamma) = \neg X_w \wedge Y_w \text{ (for a new world } w \text{ of level 0)}$$

$$p(A) = \bigwedge_{w\in W_0}(\neg X_w^{h(\neg x)} \vee \neg Y_w^{h(\neg y)}) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}(\neg X_w^{i,h(\neg x)} \vee \neg Y_w^{i,h(\neg y)})$$

$$p(\neg A) = \bigwedge_{i=1}^{2n}(X_w^i \wedge Y_w^i) \text{ (for a new world } w \text{ of level 0)}$$

$$p(B) = \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}(\neg X_w \vee Y_w^i) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\bigwedge_{j=1}^{2n}(\neg X_w^i \vee Y_w^{i,j})$$

$$p(\neg B) = X_w \wedge \neg Y_w^{h(\neg y)} \text{ (for a new world } w \text{ of level 0)}$$

$$p(C) = \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}(X_w^i \vee \neg Y_w) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\bigwedge_{j=1}^{2n}(X_w^{i,j} \vee \neg Y_w^i)$$

$$p(\neg C) = \neg X_w^{h(\neg x)} \wedge Y_w \text{ (for a new world } w \text{ of level 0)}$$

$$p(\mathbf{I}X \to X) = \bigwedge_{w\in W_0}(\neg X_w^{h(\neg x)} \vee X_w) \wedge \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}(\neg X_w^{i,h(\neg x)} \vee X_w^i)$$

$$p(\mathbf{I}X \to \mathbf{II}X) = \bigwedge_{w\in W_0}\bigwedge_{i=1}^{2n}\bigwedge_{j=1}^{2n}(\neg X_w^{h(\neg x)} \vee X_w^{i,j})$$

$$p(\neg X \to \mathbf{I}\neg X) = \bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} (X_w \vee \neg X_w^i) \wedge \bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} \bigwedge_{j=1}^{2n} (X_w^i \vee \neg X_w^{i,j})$$

$$p(X \to \neg\mathbf{I}\neg\mathbf{I}X) = \bigwedge_{w \in W_0} \bigwedge_{i=1}^{2n} (\neg X_w \vee X_w^{h(x),i})$$

Theorem 5.9 RSAT(RCC-8) *can be polynomially transformed to* SAT

Proof: Since every abbreviation, the axioms and the properties of closed regions are transformed to equivalent propositional formulas with respect to the existence of an RCC-8-model, $m(\Theta)$ is satisfiable iff $p(m(\Theta))$ is satisfiable. It remains to be proven that this is a polynomial transformation: Every world together with every region corresponds to a proposition, which are polynomial many. Every abbreviation, the axioms and formulas (21) and (22) introduce polynomial many clauses, so the size of the propositional formula obtained by the transformation of $m(\Theta)$ is polynomial in the number of regions in Θ. □

Together with Corollary 4.4 this results in the following theorem.

Theorem 5.10 RSAT(RCC-8) *is* NP*-complete.*

6 Tractable Subsets of RCC-8

In order to identify a tractable subset of RCC-8, we analyze which relations can be expressed as propositional Horn formulas, as the propositional satisfiability problem for Horn formulas (HORNSAT) is tractable [9].

Proposition 6.1 *Applying the transformation p to the model and entailment constraints, to the axioms for $\mathbf{I}$, and to the properties of closed regions as specified in Proposition 5.8 leads to Horn formulas. Formulas (21) and (22) are also Horn formulas.*

With these formulas further Horn formulas can be specified.

Lemma 6.2 *The following disjunctions of abbreviations will be transformed to propositional Horn formulas:*

$$\begin{array}{lll} \alpha \vee K & \textit{with} & K \in \{\beta, \neg\beta, \gamma, \neg\gamma, A, \neg A, B, \neg B, C, \neg C\}, \\ A \vee K & \textit{with} & K \in \{\neg\alpha, \beta, \neg\beta, \gamma, \neg\gamma, B, \neg B, C, \neg C\}. \end{array}$$

Proof: The propositional formulas resulting from the model constraints α and A are indefinite Horn formulas. Because the propositional formulas resulting from the other constraints are Horn formulas, and the disjunction of an indefinite Horn formula with another Horn formula is again a Horn formula, the lemma holds. □

All relations with an abbreviated form using only abbreviations or disjunctions of abbreviations transformable to Horn formulas can be transformed to Horn formulas. In this way 64 relations can be transformed to Horn formulas. They are listed in Appendix B.1 together with their abbreviated form. We call the subset of RCC-8 containing these relations $\mathcal{H}_8$.

Theorem 6.3

1. $\mathsf{RSAT}(\mathcal{H}_8)$ *can be polynomially reduced to* HORNSAT
2. $\mathsf{RSAT}(\widehat{\mathcal{H}}_8) \in \mathsf{P}$

Proof: Every relation of $\mathcal{H}_8$ can be written as a Horn formula. So every set Θ of RCC-8-formulas can be written as a conjunction of the propositional formulas of their elements, which is also a Horn formula. So $\mathsf{RSAT}(\mathcal{H}_8) \in \mathsf{P}$ and because of Corollary 4.7 the closure of $\mathcal{H}_8$ is also in P. □

Theorem 6.4 $\widehat{\mathcal{H}}_8$ *contains the following* 148 RCC-8 *relations:*

$$\widehat{\mathcal{H}}_8 = \mathsf{RCC\text{-}8} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$$

where

$$\begin{aligned}\mathcal{N}_3 \quad = \quad & \{R | \{\mathsf{EQ}\} \subseteq R \textit{ and } ((\{\mathsf{NTPP}\} \subseteq R, \{\mathsf{TPP}\} \not\subseteq R) \\ & \textit{or } (\{\mathsf{NTPP}^{-1}\} \subseteq R, \{\mathsf{TPP}^{-1}\} \not\subseteq R))\}.\end{aligned}$$

$\mathcal{N}_1$ *and* $\mathcal{N}_2$ *were defined in Lemma 4.12.*

For proving that $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8 we have to show that no relation of $\mathcal{N}_3$ can be added to $\widehat{\mathcal{H}}_8$ without making RSAT intractable. By computing the closure of $\widehat{\mathcal{H}}_8$ with each relation of $\mathcal{N}_3$ we get the following lemma.

Lemma 6.5 *The closure of every subset of* RCC-8 *containing* $\widehat{\mathcal{H}}_8$ *and one relation of* $\mathcal{N}_3$ *contains the relation* $\{\mathsf{EQ}, \mathsf{NTPP}\}$.

Therefore it is sufficient to prove NP-hardness of $\mathsf{RSAT}(\widehat{\mathcal{H}}_8 \cup \{\mathsf{EQ}, \mathsf{NTPP}\})$ for showing that $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8.

Theorem 6.6 $\mathsf{RSAT}(\widehat{\mathcal{H}}_8 \cup \{\mathsf{EQ}, \mathsf{NTPP}\})$ *is* NP*-complete.*

Proof: Transformation of 3SAT to RSAT($\widehat{\mathcal{H}}_8 \cup \{\mathsf{EQ}, \mathsf{NTPP}\}$). Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be a set of variables and $\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$ be a set of clauses of an arbitrary instance of 3SAT with $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$, where $l_{i,j}$ are literals over variables of $\mathcal{V}$. We will construct a set of spatial formulas Θ using only relations of $\widehat{\mathcal{H}}_8 \cup \{\mathsf{EQ}, \mathsf{NTPP}\}$, such that Θ is satisfiable iff $\mathcal{C}$ is a positive instance of 3SAT, using the following three transformation steps:

1. For each variable $v_L \in \mathcal{V}$ the regions X_L, Y_L, $X_{\neg L}$, and $Y_{\neg L}$ will be introduced by adding the following spatial formulas to Θ (see Figure 7):

$$X_L\{\mathsf{EQ}, \mathsf{NTPP}\}Y_L, X_{\neg L}\{\mathsf{EQ}, \mathsf{NTPP}\}Y_{\neg L},$$
$$X_L\{\mathsf{EC}, \mathsf{NTPP}\}X_{\neg L}, Y_L\{\mathsf{TPP}\}Y_{\neg L},$$
$$X_L\{\mathsf{TPP}, \mathsf{NTPP}\}Y_{\neg L}, Y_L\{\mathsf{EC}, \mathsf{TPP}\}X_{\neg L}.$$

2. For each literal occurrence $l_{i,j}$ the regions $X_{i,j}$ and $Y_{i,j}$ will be introduced by adding the spatial formula $X_{i,j}\{\mathsf{EQ}, \mathsf{NTPP}\}Y_{i,j}$ to Θ. Depending on whether the literal occurrence is positive or negative different spatial formulas have to be added to Θ.

   (a) $l_{i,j} = v_L$:

$$X_{i,j}\{\mathsf{EC}, \mathsf{NTPP}\}X_{\neg L}, Y_{i,j}\{\mathsf{TPP}\}Y_{\neg L},$$
$$X_{i,j}\{\mathsf{TPP}, \mathsf{NTPP}\}Y_{\neg L}, Y_{i,j}\{\mathsf{EC}, \mathsf{TPP}\}X_{\neg L}.$$

   (b) $l_{i,j} = \neg v_L$:

$$X_{i,j}\{\mathsf{EC}, \mathsf{NTPP}\}X_L, Y_{i,j}\{\mathsf{TPP}\}Y_L,$$
$$X_{i,j}\{\mathsf{TPP}, \mathsf{NTPP}\}Y_L, Y_{i,j}\{\mathsf{EC}, \mathsf{TPP}\}X_L.$$

3. For each clause $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$ the following spatial formulas will be added to Θ:

$$Y_{i,1}\{\mathsf{NTPP}^{-1}\}X_{i,2}, Y_{i,2}\{\mathsf{NTPP}^{-1}\}X_{i,3}, Y_{i,3}\{\mathsf{NTPP}^{-1}\}X_{i,1}.$$

With this construction for every variable as well as for every literal occurrence two regions X and Y (with the appropriate indices) are introduced. When a literal occurrence or a variable is true, the corresponding regions hold the relation $X\{\mathsf{NTPP}\}Y$. When it is false, they hold the relation $X\{\mathsf{EQ}\}Y$. Transformation step 1 introduces the regions corresponding to the positive and the negative literal of each variable. This construction assures that both literals have complementary assignments. This is shown in Figure 7, (a) is the original configuration, (b) and (c) are the only possible refinements of all relations to base relations. Transformation step 2 assures that regions of literal occurrences hold the same relation as the regions of the corresponding variables. Finally, transformation step 3 makes sure that at least one literal occurrence of every

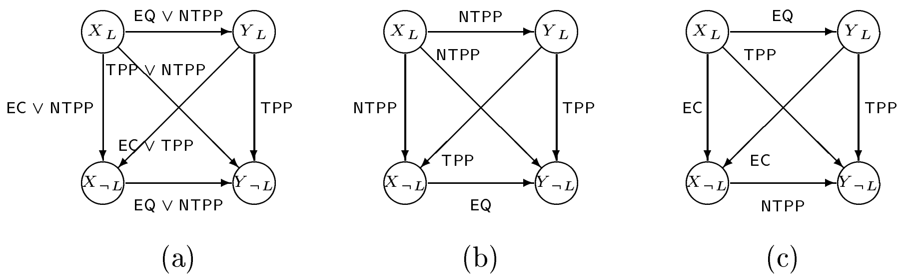

Figure 7: Construction for complementary literals in the proof of Theorem 6.6.

clause is true. If all literal occurrences of a clause are false, the corresponding regions hold the relation $\{\mathsf{EQ}\}$. Then there is a path starting at $X_{i,1}$ passing $X_{i,2}$ and $X_{i,3}$ ending at $X_{i,1}$, where $\{\mathsf{NTPP}^{-1}\}$ and $\{\mathsf{EQ}\}$ are the only occurring relations, which is inconsistent. All other combinations are possible.

RSAT $\Rightarrow$ 3SAT: Suppose that the CSP A obtained by transformation from a given instance B of 3SAT is consistent, and suppose that α is a consistent instantiation of A. Then an assignment β that satisfies B can easily be obtained in the following way: For every variable $v_L \in \mathcal{V}$, if $X_L\{\mathsf{NTPP}\}Y_L$ holds, then $\beta(v_L)$ is true, otherwise $\beta(v_L)$ is false.

3SAT $\Rightarrow$ RSAT: Suppose that B is a positive instance of 3SAT and β is an assignment that satisfies B. Then the CSP A obtained by transformation from B is consistent. We will show this by constructing a spatial configuration M that holds all relations of A.

For every literal and every literal occurrence that is false, the corresponding regions hold the relation EQ and can therefore be treated as a single region. Figure 8(a) shows the regions corresponding to a variable P with $\beta(P) = \text{true}$ (placed inside the dashed box) and regions corresponding to a positive and a negative literal occurrence of P. Figure 8(b) shows the same for a variable Q where $\beta(Q) = \text{false}$. For each variable there might be different corresponding literal occurrences, but all hold the same constraints. The NTPP relations in Figure 8 pointing to or from nowhere indicate the possible constraints required by transformation step 3. It can be seen that for each variable $V \in \mathcal{V}$ region $Y_{\neg V}$ contains all other regions of the same variable and of the corresponding literal occurrences.

For constructing a spatial configuration we will use m different regions C_i, one for each clause $c_i \in \mathcal{C}$, with $C_i\{\mathsf{DC}\}C_j$ for every i, j. All regions corresponding to literal occurrences of a clause c_i are arranged consistently, i.e. holding the relations of transformation step 3, inside the region C_i. The spatial regions of M corresponding to the same variable consist of m different parts, each part contained in a different region C_i. These parts are arranged inside C_i in a way that they hold the relations among each other as well as the relations to the

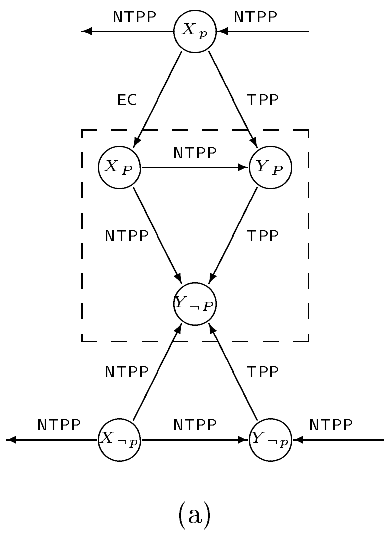

Figure 8: Regions corresponding to a variable and their literal occurrences.

regions corresponding to the literal occurrences according to the transformation. As the regions corresponding to the literal occurrences of a clause c_i either hold the relations $\{\mathsf{TPP}\}, \{\mathsf{NTPP}\}$ or $\{\mathsf{EC}\}$ to the parts of the regions corresponding to the variables inside C_i, they also hold these relations to the complete regions corresponding to the variables. Therefore M is a consistent model for A.

The specified transformation takes linear time in the number of clauses, so it has been proven that $\mathsf{RSAT}(\widehat{\mathcal{H}}_8 \cup \{\mathsf{EQ}, \mathsf{NTPP}\})$ is NP-hard. Because of Corollary 5.10 it is also NP-complete. $\square$

Lemma 6.5 together with Theorem 6.6 results in the following theorem.

Theorem 6.7 *$\widehat{\mathcal{H}}_8$ is a maximal tractable subset of* RCC-8.

It has to be noted that there might be other maximal subsets of RCC-8 that contain all base relations.

As $\widehat{\mathcal{H}}_8$ is a tractable subset of RCC-8, the intersection of RCC-5 and $\widehat{\mathcal{H}}_8$ is also tractable. We will call this subset $\widehat{\mathcal{H}}_5$.

Proposition 6.8 *$\widehat{\mathcal{H}}_5$ contains all relations of* RCC-5 *except for the relations* $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$, $\{\mathsf{DR}, \mathsf{PP}, \mathsf{PP}^{-1}\}$, $\{\mathsf{PP}, \mathsf{PP}^{-1}, \mathsf{EQ}\}$ *and* $\{\mathsf{DR}, \mathsf{PP}, \mathsf{PP}^{-1}, \mathsf{EQ}\}$.

The following lemma can easily be obtained by computing the closure of the employed sets.

Lemma 6.9 *The closure of the set of all* RCC-5 *base relations together with one of the relations* $\{\mathsf{DR}, \mathsf{PP}, \mathsf{PP}^{-1}\}$, $\{\mathsf{PP}, \mathsf{PP}^{-1}, \mathsf{EQ}\}$ *or* $\{\mathsf{DR}, \mathsf{PP}, \mathsf{PP}^{-1}, \mathsf{EQ}\}$ *contains* $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$.

Theorem 6.10 *$\widehat{\mathcal{H}}_5$ is the only maximal tractable subset of* RCC-5 *containing all base relations.*

Proof: $\widehat{\mathcal{H}}_5$ is by definition a tractable subset of RCC-5. To prove NP-hardness of RCC-5 the relations $\{\mathsf{PO}\}$, $\{\mathsf{PP}, \mathsf{PP}^{-1}\}$ and $\{*\}$ were used in Lemma 4.3. With Lemma 6.9 it follows that $\widehat{\mathcal{H}}_5$ contains all relations except for those making a set containing all base relations NP-complete. □

7 Applicability of Path-Consistency

In the previous section we proved that $\widehat{\mathcal{H}}_8$ is a tractable subset of RCC-8. This means that RSAT($\widehat{\mathcal{H}}_8$) can be solved in polynomial time. In our case this is done by first transforming a set of $\widehat{\mathcal{H}}_8$ formulas to a propositional Horn formula. The resulting Horn formula can then be solved in time linear in the number of literals [5]. Because the number of literals is considerably higher than n^3, this way of solving RSAT does not appear to be very efficient.

One popular algorithm for solving CSP's with a running time of $O(n^3)$ is the previously mentioned path-consistency method, but this method is not complete in general. In this section we will first proof prove that the path-consistency method is sufficient for deciding consistency of RSAT($\mathcal{H}_8$) and based on this that it is also sufficient for deciding consistency of RSAT($\widehat{\mathcal{H}}_8$). This is done by showing that the path-consistency method finds an inconsistency whenever positive unit resolution (PUR) resolves the empty clause from the corresponding propositional formula. As PUR is refutation-complete for Horn formulas, it follows that the path-consistency method decides RSAT($\mathcal{H}_8$).

The only way to get the empty clause using PUR is resolving a positive and a negative unit clause of the same variable. Since the Horn formulas that are used contain only a few types of different clauses, there are only a few ways to resolve unit clauses using PUR. Before we proceed with the actual proof, we have to define a few terms and state some lemmata.

Definition 7.1

- *R_K denotes the set of relations of $\mathcal{H}_8$ with the conjunct K appearing in their abbreviated form (see Appendix B). $R_K(X, Y)$ means that the relation holding between X and Y is in R_K.*
- *R_{K_1,K_2} denotes $R_{K_1} \cup R_{K_2}$*
- *An R_K-chain $R^*_K(X, Y)$ is a path from region X to region Y, where all relations between successive regions are from R_K.*
- *R_{σ^*} is written instead of $R_{\sigma,\alpha\vee\sigma,A\vee\sigma}$ for any abbreviation σ.*

The following proposition contains some important observations made from the transformation of Θ to propositional logic as specified in Section 5. In the following $\{X^*_w\}$ denotes either $\{X_w\}$, $\{X^i_w\}$ or $\{X^{i,j}_w\}$.

Proposition 7.2 *Let Θ be a set of $\mathcal{H}_8$-formulas. Then $p(m(\Theta))$ is the corresponding Horn formula and $c(\Theta)$ the corresponding set of Horn clauses. The following results from the transformation of the abbreviated form of Θ to $p(m(\Theta))$:*

1. *For every world w at most two unit input clauses are contained in $c(\Theta)$. At least one of them is positive.*
2. *If the unit clauses $\{X_w\}$ and $\{Y_w\}$ or $\{X^i_w\}$ and $\{Y^i_w\}$ are input clauses of $c(\Theta)$, then $R_{\neg\alpha}(X,Y)$ or $R_{\neg A}(X,Y)$ holds, respectively.*
3. *If the unit clause $\{\neg X_w\}$ or $\{\neg X^i_w\}$ is an input clause of $c(\Theta)$, there is a region $Y \in Reg(\Theta)$ with $R_{\neg\gamma}(X,Y)$ or $R_{\neg C}(X,Y)$, respectively. Then $\{Y_w\}$ is also an input clause of $c(\Theta)$.*
4. *If none of the unit clauses $\{X^*_w\}$ is input clause and one of them is resolved from $c(\Theta)$, there is a region $Y \in Reg(\Theta)$ with $R_{\gamma^*,C^*}(X,Y)$ and the corresponding unit clause $\{Y^*_w\}$ must be present. The relation R_{C^*} can only be used for the unit clauses $\{X^i_w\}$ and $\{X^{i,j}_w\}$.*
5. *If one of the unit clauses $\{\neg X^*_w\}$ is resolved from $c(\Theta)$, there is a region $Y \in Reg(\Theta)$ with $R_{\alpha,A}(X,Y)$ and the corresponding unit clause $\{Y^*_w\}$ must be present. The relation R_A can only be used for the unit clauses $\{\neg X^i_w\}$ and $\{\neg X^{i,j}_w\}$.*
6. *If a unit clause of the type $\{X^{h(\neg z)}_w\}$ for a particular Z was first resolved for region X, there is a region $Y \in Reg(\Theta)$ with $R_C(X,Y)$.*
7. *If one of the clauses $\{X^i_w\}$ or $\{X^{i,j}_w\}$ is present, the clause $\{X_w\}$ can be resolved.*

As shown in Appendix B, the sets of relations used in Proposition 7.2 contain the following relations:

$$\begin{aligned}
R_\gamma &= \{\{\mathsf{TPP}^{-1}\},\{\mathsf{NTPP}^{-1}\},\{\mathsf{EQ}\},\{\mathsf{TPP}^{-1},\mathsf{NTPP}^{-1}\},\\
&\qquad \{\mathsf{TPP}^{-1},\mathsf{NTPP}^{-1},\mathsf{EQ}\}\},\\
R_{\alpha\vee\gamma} &= \{R\cup\{\mathsf{DC}\}|R\in R_\gamma\},\\
R_{A\vee\gamma} &= \{R\cup\{\mathsf{EC}\}|R\in(R_\gamma\cup R_{\alpha\vee\gamma})\},\\
R_C &= \{\{\mathsf{NTPP}^{-1}\}\},\\
R_{\alpha\vee C} &= \{\{\mathsf{DC},\mathsf{NTPP}^{-1}\}\},\\
R_{A\vee C} &= \{\{\mathsf{EC},\mathsf{NTPP}^{-1}\},\{\mathsf{DC},\mathsf{EC},\mathsf{NTPP}^{-1}\}\},\\
R_\alpha &= \{\{\mathsf{DC}\}\},\\
R_A &= \{\{\mathsf{DC}\},\{\mathsf{EC}\},\{\mathsf{DC},\mathsf{EC}\}\}.
\end{aligned}$$

It can be seen, that $R_C \subseteq R_\gamma$, $R_{\alpha\vee C} \subseteq R_{\alpha\vee\gamma}$ and $R_{A\vee C} \subseteq R_{A\vee\gamma}$, so, for example, R_{γ^*} can be written instead of R_{γ^*,C^*}.

With PUR, positive unit clauses can only be resolved in a very specific way. This is described in the following lemma.

Lemma 7.3 *Let the clauses $\{Y_w^*\}$ and $\{Z_w^*\}$ be input clauses of $c(\Theta)$ and the clauses $\{X_w^*\}$ be no input clauses of $c(\Theta)$. (If there is only one positive input clause of world w, then $Y \equiv Z$). The positive unit clause $\{X_w^*\}$ can only be resolved with PUR using an R_{γ^*}-chain from X to Y or an R_{γ^*}-chain from X to Z.*

Proof: According to the Proposition 7.2, item 4, there is a region Z^1 with $R_{\gamma^*}(X, Z^1)$ and the corresponding unit clause $\{Z_w^{1*}\}$ must be present. If $Z^1 \not\equiv Y$ and $Z^1 \not\equiv Z$ this clause cannot be an input clause so there must be another region Z^2 with $R_{\gamma^*}(Z^1, Z^2)$ and so on until $Z^n \equiv Y$ or $Z^n \equiv Z$ for a region Z^n. □

The resolution of a positive unit clause using an R_{γ^*}-chain has some side effects on the relations contained in the chain.

Lemma 7.4 *If one of the positive unit clauses $\{X_w^*\}$ is resolved using PUR, then every relation of the required R_{γ^*}–chain will be reduced to a relation of R_γ.*

Proof: Let Y and Z be two successive regions of the R_{γ^*}–chain, holding either $R_{\alpha\vee\gamma}(Y, Z)$ or $R_{A\vee\gamma}(Y, Z)$. Then the required unit clause $\{Y_w^*\}$ resolves from a clause of the type $\{Y_w^*, \neg Z_w^*, \neg Y_u^*, \neg Z_u^*\}$, which consists of two disjunctively connected parts, the α or A part $\{\neg Y_u^*, \neg Z_u^*\}$ and the γ part $\{Y_w^*, \neg Z_w^*\}$. For this resolution the unit clauses $\{Y_u^*\}$ and $\{Z_u^*\}$ are needed which are inconsistent with $R_{\alpha\vee A}(Y, Z)$. □

For the R_{γ^*}-chain used to resolve a particular positive unit clause other positive unit clauses are necessary for which other R_{γ^*}-chains might be used. In order to refer to all R_{γ^*}-chains that are used to resolve a particular positive unit clause we introduce the notion of chain structure.

Definition 7.5 *Let R be an $R_{\alpha\vee\gamma,A\vee\gamma}$-relation of an R_{γ^*}-chain.*

- *The chain structure of R contains all R_{γ^*}-chains used to reduce R to a relation of R_γ.*
- *Let $\prec$ be a relation on chain structures. Let S_1 be the chain structure of $R_1(X^1, Y^1)$ and S_2 be the chain structure of $R_2(X^2, Y^2)$. $S_1 \prec S_2$ holds, iff $R_1(X^1, Y^1)$ occurs in S_2.*

For some of our proofs we will use Noetherian induction on chain structures, which is defined on well-founded relations.

Definition 7.6 *A relation $\prec$ on a set M is well-founded, iff every non empty subset of M has a minimal element, i.e.* $\forall S \subseteq M.(S \neq \emptyset \rightarrow \exists x \in S.(\neg \exists y \in S.y \prec x))$.

Theorem 7.7 (Noetherian Induction) *Let $\prec$ be a well-founded relation on a set M. To prove a property P for all $x \in M$, it suffices to prove:*

$$\forall x \in M. (\forall y \in M. (y \prec x) \rightarrow P(y)) \rightarrow P(x)$$

Proof: Suppose that the set $A \subseteq M$ of elements not holding P is not empty. Then A has a minimal element m, i.e. $P(y)$ holds for all $y \prec m$. Then $P(m)$ also holds, which contradicts the assumption. So A must be empty, i.e. P holds for all $x \in M$. □

Lemma 7.8 *$\prec$ is a well-founded relation.*

Proof: If S is a chain structure where only relations of R_γ occur, then there is no chain structure S' with $S' \prec S$.
Suppose that S is the chain structure of $R(X,Y)$ and $R(X,Y)$ appears in S. If the occurrence of $R(X,Y)$ is recursive in S, S can not be used to reduce R to a relation of R_γ, so S is no chain structure in our sense. If the occurrence of $R(X,Y)$ is not recursive in S, S can be replaced by S', the chain structure belonging to the second occurrence of $R(X,Y)$.
There is only a finite number of regions, so every non empty set of chain structures has at least one minimal element. □

We have to prove that whenever PUR reduces a relation R of an R_{γ^*}-chain to one of R_γ in the sense of Lemma 7.4, the path consistency method also results in R_γ, i.e. the base relations $\{\mathsf{DC}\}$ and $\{\mathsf{EC}\}$ will be excluded from R. For this we need the following operations which can be verified using Table 1.

Proposition 7.9 *Let R be a relation of* RCC-8.

1. *R_γ is closed under composition.*
2. *$R_C \circ R_\gamma = R_C$ and $R_\gamma \circ R_C = R_C$*
3. *If $\{\mathsf{DC}, \mathsf{EC}\} \cap R = \emptyset$, then $\{\mathsf{DC}, \mathsf{EC}\} \cap (R_\gamma \circ R) = \emptyset$ and $\{\mathsf{DC}, \mathsf{EC}\} \cap (R \circ R_\gamma^\smile) = \emptyset$.*
4. $\{\mathsf{DC}, \mathsf{EC}\} \cap (R_\gamma \circ R_\gamma^\smile) = \emptyset$.
5. $\{\mathsf{DC}\} \cap (R_\gamma \circ \{\mathsf{EC}\} \circ R_\gamma^\smile) = \emptyset$.
6. $\{\mathsf{DC}, \mathsf{EC}\} \cap (R_C \circ \{\mathsf{EC}\} \circ R_\gamma^\smile) = \{\mathsf{DC}, \mathsf{EC}\} \cap (R_\gamma \circ \{\mathsf{EC}\} \circ R_C^\smile) = \emptyset$

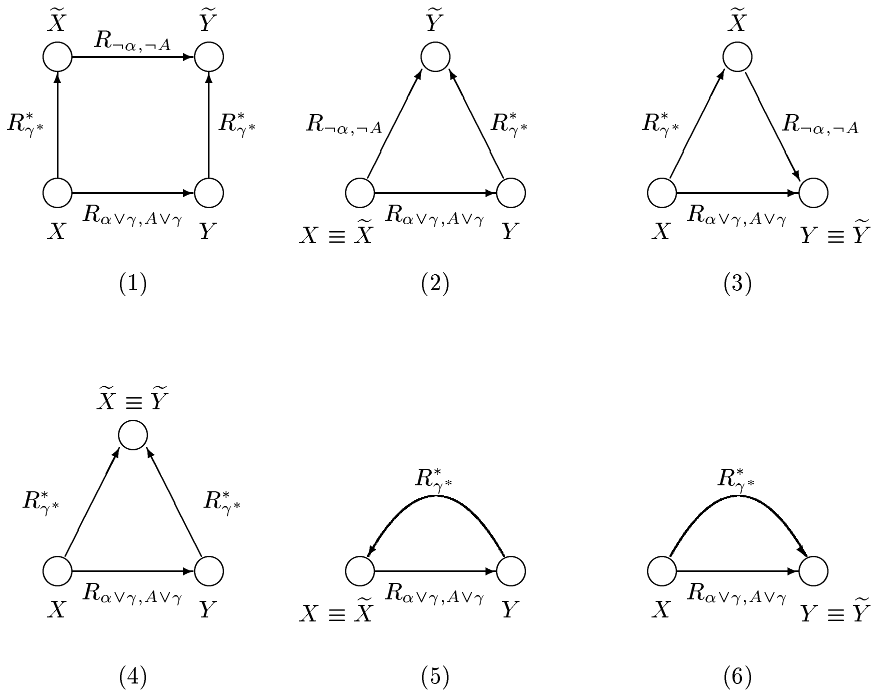

Figure 9: The six possible cases of the proof to Lemma 7.10. In the lower row $\widetilde{X}$ is equal to $\widetilde{Y}$.

Lemma 7.10 *If PUR reduces every relation of an R_{γ^*}-chain K from X^1 to Y^1 to a relation of R_γ, the path-consistency method results in $R_\gamma(X^1, Y^1)$.*

Proof: Let $P(S) \equiv$ "If PUR reduces an R_{γ^*}-relation R to one of R_γ using the chain structure S, the path-consistency method also reduces R to one of R_γ". We will prove $P(S)$ with Noetherian induction.

Induction hypothesis: $\forall S' \prec S.\quad P(S')$.

Suppose that PUR reduces R to one of R_γ to obtain the clause $\{X^*_w\}$. If R is already in R_γ nothing has to be proven. If R is one of $R_{\alpha\vee\gamma,A\vee\gamma}$, the clause $\{X^*_w, \neg Y^*_w, \neg X^*_u, \neg Y^*_u\}$ is input clause of $c(\Theta)$. If $R_{\alpha,A}(X,Y)$ can be excluded with PUR, the unit clauses $\{X^*_u\}$ and $\{Y^*_u\}$ must be present. Let the clauses $\{\widetilde{X}^*_u\}$ and $\{\widetilde{Y}^*_u\}$ be the only unit input clauses of the world u, introduced by either $R_{\neg\alpha}(\widetilde{X}, \widetilde{Y})$ or $R_{\neg A}(\widetilde{X}, \widetilde{Y})$. Then there are R_{γ^*}-chains from X to $\widetilde{X}$ and from Y to $\widetilde{Y}$ that are part of S. Six different cases, shown in Figure 9, must be distinguished:

Case 1: A chain structure $S' \prec S$ belongs to any relation $R' \in R_{\alpha\vee\gamma,A\vee\gamma}$ of the R_{γ^*}-chains from X to $\widetilde{X}$ and from Y to $\widetilde{Y}$. Since $P(S')$ holds by induction hypothesis, and R_γ is closed under composition, $R_\gamma(X, \widetilde{X})$ and $R_\gamma(Y, \widetilde{Y})$ are obtained by the path-consistency method.

a. If $R \in R_{\alpha\vee\gamma}$, then R is reduced to one of R_γ by the path-consistency method, because of items 3 and 5 of Proposition 7.9.

b. If $R \in R_{A\vee\gamma}$ then $\{X_w^{h(\neg x)}\}$ and $\{Y_w^{h(\neg y)}\}$ must be obtained, so the R_{γ^*}-chains from X to $\widetilde{X}$ and from Y to $\widetilde{Y}$ must each contain at least one relation of R_C. Because of item 2 of Proposition 7.9 the path-consistency method results in $R_C(X, \widetilde{X})$ and $R_C(Y, \widetilde{Y})$. Because of items 3 and 6 it results in $R_\gamma(X, Y)$.

The remaining five cases can be handled in the same way as case 1. All the R_{γ^*}-chains can be reduced to R_γ using the induction hypothesis. With the operations of Proposition 7.9 the path-consistency method reduces R to one of R_γ.

With this $P(S)$ has been proven for all chain structures S. Every relation of the R_{γ^*}-chain K can now be reduced to one of R_γ with the path-consistency method. Because R_γ is closed under composition, the path-consistency method results in $R_\gamma(X^1, Y^1)$. □

Theorem 7.11 *The path-consistency method decides* $\mathsf{RSAT}(\mathcal{H}_8)$.

Proof: Let Θ be an inconsistent set of $\mathcal{H}_8$ formulas, $p(m(\Theta))$ the equivalent propositional Horn formula and $c(\Theta)$ the corresponding set of clauses. Since Θ is inconsistent, the empty clause can be resolved from $c(\Theta)$ using PUR.

1. The empty clause is resolved from $\{X_w\}$ and $\{\neg X_w\}$.

   (a) $\{\neg X_w\}$ is input clause of $c(\Theta)$. Then, with Proposition 7.2, $\{Y_w\}$ is also an input clause for some region Y with $R_{\neg\gamma}(X, Y)$. $\{X_w\}$ is resolved from $c(\Theta)$, so there must be an R_{γ^*}-chain from X to Y. Because of Lemma 7.10, the path-consistency method results in $R_\gamma(X, Y)$, which is a contradiction to $R_{\neg\gamma}(X, Y)$.

   (b) $\{X_w\}$ is the only positive input clause of $c(\Theta)$ of world w. Then $\{\neg X_w\}$ is resolved from $c(\Theta)$. According to Proposition 7.2, there must be a region Y with $R_{\alpha,A}(X, Y)$ and $\{Y_w\}$ must be present. $\{Y_w\}$ is resolved using an R_{γ^*}-chain from Y to X, which is reduced to $R_\gamma(Y, X)$ by the path-consistency method. This is a contradiction to $R_{\alpha,A}(X, Y)$.

   (c) $\{X_w\}$ is an input clause of $c(\Theta)$ and $\{Z_w\}$ is also an input clause for some region Z. According to Proposition 7.2, $R_{\neg\alpha}(X, Z)$ must hold. $\{\neg X_w\}$ is resolved from $c(\Theta)$, so there must be a region Y with $R_{\alpha,A}(X, Y)$ and $\{Y_w\}$ must also be resolved from $c(\Theta)$. So there is either an R_{γ^*}-chain from Y to X, which is equal to (b), or an R_{γ^*}-chain from Y to Z which is reduced to $R_\gamma(Y, Z)$ by the path-consistency method. If $R_\alpha(X, Y)$ holds, the path-consistency method results in an inconsistency. If $R_A(X, Y)$ holds, then one relation of the R_{γ^*}-chain from Y to Z must be from R_C, so the path consistency method results in $R_C(Y, Z)$, which is also inconsistent.

(d) Neither $\{X_w\}$ nor $\{\neg X_w\}$ are input clauses of $c(\Theta)$, so both are resolved using PUR. The negative clause can only be resolved if there is a region Y with $R_{\alpha,A}(X,Y)$ and $\{Y_w\}$ can also be resolved. As it was shown in the six cases of Lemma 7.10, $\{X_w\}$ and $\{Y_w\}$ can only be resolved from $c(\Theta)$ if the path-consistency method results in an inconsistency with $R_{\alpha,A}(X,Y)$.

2. The empty clause is resolved from $\{X_w^i\}$ and $\{\neg X_w^i\}$. Only $\{X_w^i\}$ or $\{\neg X_w^{h(\neg x)}\}$ can be input clauses of $c(\Theta)$.

   (a) The empty clause is resolved from $\{X_w^{h(\neg x)}\}$ and the input clause $\{\neg X_w^{h(\neg x)}\}$. According to Proposition 7.2 there is a region Y with $R_{\neg C}(X,Y)$ and $\{Y_w\}$ is also input clause. The clause $\{X_w^{h(\neg x)}\}$ is resolved using an R_{γ^*}-chain from X to Y. Because of Proposition 7.2, one of the relations of the chain must be in R_C, so the path consistency method results in $R_C(X,Y)$ which contradicts $R_{\neg C}(X,Y)$.

   (b) $\{X_w^i\}$ is input clause, then there is a region Y with $R_{\neg A}(X,Y)$ and $\{Y_w^i\}$ is also an input clause of $c(\Theta)$. To resolve $\{\neg X_w^i\}$, there must be a region Z with $R_{\alpha,A}(X,Z)$ and $\{Z_w^i\}$ must be present. $\{Z_w^i\}$ can be resolved using an R_{γ^*}-chain from Y to Z. The path-consistency method results in $R_\gamma(Y,Z)$ which is, composed with $R_{\neg A}(X,Y)$, inconsistent with $R_{\alpha,A}(X,Z)$.

   (c) Neither $\{X_w^i\}$ nor $\{\neg X_w^i\}$ are input clauses and must be resolved from $c(\Theta)$. Then there must be a region Y with $R_{\alpha,A}(X,Y)$ and $\{Y_w^i\}$ must be present. To obtain $\{X_w^i\}$ and $\{Y_w^i\}$ the six cases of Lemma 7.10 must be considered, where $R_{\alpha,A}(X,Y)$ holds instead of $R_{\gamma^*}(X,Y)$. Since $R_{\alpha,A}$ could be excluded in Lemma 7.10, it can also be excluded now, so the path-consistency method finds the inconsistency.

3. The empty clause is resolved from $\{X_w^{i,j}\}$ and $\{\neg X_w^{i,j}\}$ and neither one of the clauses can be an input clause of $c(\Theta)$. $\{\neg X_w^{i,j}\}$ can only be resolved if there is a region Y with $R_A(X,Y)$ and $\{Y_w^{i,j}\}$ must be present. To obtain $\{X_w^{i,j}\}$ and $\{Y_w^{i,j}\}$ the six cases of Lemma 7.10 must be considered, where $R_A(X,Y)$ holds instead of $R_{\gamma^*}(X,Y)$. Since R_A could be excluded in Lemma 7.10, it can also be excluded now, so the path-consistency method finds the inconsistency.

So whenever PUR finds an inconsistency it is also found by the path-consistency method. Since positive unit resolution is refutation-complete for propositional Horn formulas, the path-consistency method is sufficient for deciding consistency of $\mathsf{RSAT}(\mathcal{H}_8)$. □

The path-consistency method can be used to decide $\mathsf{RSAT}(\widehat{\mathcal{H}}_8)$ when every relation of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ is transformed to relations of $\mathcal{H}_8$ according to Theorem 4.6. As this way of deciding $\mathsf{RSAT}(\widehat{\mathcal{H}}_8)$ is pretty awkward, we will prove that the path-consistency method can decide $\mathsf{RSAT}(\widehat{\mathcal{H}}_8)$ directly without any preprocessing.

This will be shown in the same way as it was shown for $\mathcal{H}_8$, namely, that the path-consistency method finds an inconsistency whenever PUR finds one. For this we need a way of transforming the relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ to propositional Horn formulas.

Proposition 7.12 *Let $\mathcal{S}$ be the set of relations S which can be transformed to a propositional Horn formula $p'(S(X,Y))$, where $p'(S(X,Y)) = p(m(S(X,Y)))$ if $S \in \mathcal{H}_8$. Since any relation of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ can be obtained by composition, converse or intersection of relations of $\mathcal{S}$, the propositional Horn formula of $R(X,Y)$ where $R \in \widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ can be obtained using this composition inductively:*

1. *If $R = S \circ T$, where $S,T \in \mathcal{S}$ and $R \notin \mathcal{S}$, then introduce a pseudo region Z that is only related to regions X and Y, holding $S(X,Z)$ and $T(Z,Y)$. Therefore $p'(R(X,Y)) = p'(S(X,Z)) \wedge p'(T(Z,Y))$. Add R to $\mathcal{S}$.*
2. *If $R = S \cap T$, where $S,T \in \mathcal{S}$ and $R \notin \mathcal{S}$, then $p'(R(X,Y)) = p'(S(X,Y)) \wedge p'(T(X,Y))$. Add R to $\mathcal{S}$.*
3. *If $R = S^{\smile}$, where $S \in \mathcal{S}$ and $R \notin \mathcal{S}$, then $p'(R(X,Y)) = p'(S(Y,X))$. Add R to $\mathcal{S}$.*

For proving that the path-consistency method decides $\mathsf{RSAT}(\mathcal{H}_8)$, the R_{γ^*}-chain was the central part as it was the only way to resolve a positive unit clause. Since some relations of $\widehat{\mathcal{H}}_8$ are combined using relations of R_{γ^*}, these relations can now also be considered for the R_{γ^*}-chain which will therefore be denoted as *extended R_{γ^*}-chain*. As all of these relation must be analyzed separately and some relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ can be obtained using different combinations, only relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ that can not be obtained without using relations of R_{γ^*} will be considered for extended R_{γ^*}-chains. In Appendix B.2 we give a list of all relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ and how they can be composed of $\mathcal{H}_8$-relations. We have chosen a composition of all relations such that the following analysis is as simple as possible. The set of relations that can be part of an extended R_{γ^*}-chain will be denoted by R_e.

Lemma 7.13 *Additional to R_{γ^*}, R_e contains the following relations:*

$$\begin{aligned}
\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{NTPP},\mathsf{TPP}^{-1},\mathsf{NTPP}^{-1}\} &= \{\mathsf{EC},\mathsf{NTPP}\} \circ \{\mathsf{DC},\mathsf{EQ}\},\\
\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{NTPP},\mathsf{NTPP}^{-1}\} &= \{\mathsf{DC},\mathsf{EQ}\} \circ \{\mathsf{EC},\mathsf{NTPP}^{-1}\},\\
\{\mathsf{EC},\mathsf{PO},\mathsf{EQ},\mathsf{TPP}^{-1},\mathsf{NTPP}^{-1}\} &= \{\mathsf{EQ},\mathsf{TPP}^{-1},\mathsf{NTPP}^{-1}\} \circ \{\mathsf{EC},\mathsf{EQ}\},\\
\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{EQ},\mathsf{TPP}^{-1},\mathsf{NTPP}^{-1}\} &= \{\mathsf{EC},\mathsf{EQ}\} \circ \{\mathsf{DC},\mathsf{EQ}\},
\end{aligned}$$

as well as the intersection of these relations with other relations of $\widehat{\mathcal{H}}_8$.

Proof: This can be proven by first computing the closure of $\mathcal{H}_8 \setminus R_{\gamma^*}$ under composition and intersection. All relations not contained in this closure can be composed by intersection of these relations with either the four specified relations or the relations of R_{γ^*}. $\square$

Definition 7.14 *Let R be a relation of an R_e-chain.*

1. *The extended chain structure of R contains all R_e-chains used to reduce R to a relation of R_γ.*
2. *Let $\prec_e$ be a relation on extended chain structures. Let T_1 be the extended chain structure belonging to $R_1(X^1, Y^1)$ and T_2 be the extended chain structure belonging to $R_2(X^2, Y^2)$. $T_1 \prec_e T_2$ holds, iff $R_1(X^1, Y^1)$ occurs in T_2.*

Lemma 7.15 *$\prec_e$ is well-founded.*

Proof: This can be proven in the same way as Lemma 7.8. □

Lemma 7.16 *If PUR reduces every relation of an R_e-chain K from X^1 to Y^1 to a relation of R_γ, the path-consistency method results in $R_\gamma(X^1, Y^1)$.*

Proof: $P(T) \equiv$ "If PUR reduces an R_e-relation R to one of R_γ using the chain structure T, the path-consistency method also reduces R to one of R_γ". We will prove this by Noetherian induction.

Induction hypothesis: $\forall T' \prec T. \quad P(T')$

Suppose that PUR reduces R to one of R_γ to obtain the clause $\{X_w^*\}$.

1. $R \in R_{\gamma^*}$: The six different cases of Lemma 7.10 can also be applied here. Because of the induction hypothesis, all the required extended R_{γ^*}-chains will be reduced to R_γ with the path-consistency method. Therefore R will also be reduced to R_γ with the path-consistency method.
2. $R = \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{NTPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$: Only the second relation of the composition of R is in R_{γ^*}, so $\{X_w^*\}$ can not be obtained with this relation.
3. $R = \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{NTPP}^{-1}\}$: Suppose this relation holds between the regions X and Y with the pseudo region Z. Then clauses of the type $\{\neg X_w^*, \neg Z_w^*, \neg X_v^*, Z_v^*\}$, $\{\neg X_w^*, \neg Z_w^*, X_v^*, \neg Z_v^*\}$, $\{\neg Z_w^{*,h(\neg z)}, \neg Y_w^{*,h(\neg y)}, Z_v^*, \neg Y_v^*\}$ and $\{\neg Z_w^{*,h(\neg z)}, \neg Y_w^{*,h(\neg y)}, Z_v^{k,*}, \neg Y_v^*\}$ for all $w, v \in W_0$ as well as $\{Z_i\}$, $\{Y_i\}$, $\{Z_j\}$ and $\{\neg Y_j\}$ for some i, j are input clauses. To resolve $\{X_w^*\}$, the unit clauses $\{Z_w^*\}$, $\{X_u^*\}$, and $\{Z_u^*\}$ are necessary for some u. How can $\{Z_u^*\}$ be resolved?
   (a) If $\{Z_u\}$ is the only positive input clause of world u, then $\{X_u\}$ is resolved using $\{Z_v^*\}$ and $\{X_v^*\}$ for some $v \neq u$. Then $\{X_w^*\}$ can also be resolved using $\{Z_v^*\}$ and $\{X_v^*\}$, which contradicts the assumption that it is resolved using $\{Z_u\}$ and $\{X_u\}$.

(b) If $\{Z_u\}$ and $\{Y_u\}$ are input clauses, there might be an additional extended R_{γ^*}-chain from X to Y, not passing Z. With the induction hypothesis $R_\gamma(X,Y)$ will be obtained with the path-consistency method, if it also results from PUR.

(c) If $\{Z_u^*\}$ is no input clause, there is a region $\widetilde{Z}_u$ where the world u is introduced. If the extended R_{γ^*}-chain from Z to $\widetilde{Z}_u$ passes X, the clauses $\{Z_v^*\}$ and $\{X_v^*\}$ are necessary for some $v \neq u$. This is again a contradiction to our assumption. If the chain passes Y, then the clauses $\{Y_u^*\}$, $\{Z_v^{*,h(\neg z)}\}$ and $\{Y_v^{*,h(\neg y)}\}$ are necessary for some v. To resolve $\{Z_v^{*,h(\neg z)}\}$, the clauses $\{Y_v^{*,h(\neg z)}\}$, $\{Z_s^{*,h(\neg z)}\}$, and $\{Y_s^{*,h(\neg y)}\}$ are necessary for some $s \neq v$. Then $\{Z_u^*\}$ can also be resolved using $\{Z_s^{*,h(\neg z)}\}$ and $\{Y_s^{*,h(\neg y)}\}$, which contradicts the assumption that it is resolved using $\{Z_v^{*,h(\neg z)}\}$ and $\{Y_v^{*,h(\neg y)}\}$.

4. $R = \{\mathsf{EC}, \mathsf{PO}, \mathsf{EQ}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$: Suppose this relation holds between the regions X and Y with the pseudo region Z, then clauses of the type $\{X_w^*, \neg Z_w^*\}$, $\{\neg Z_w^{*,h(\neg z)}, \neg Y_w^{*,h(\neg y)}, \neg Z_v^*, Y_v^*\}$, and $\{\neg Z_w^{*,h(\neg z)}, \neg Y_w^{*,h(\neg y)}, Z_v^*, \neg Y_v^*\}$ for all $w, v \in W_0$ as well as $\{X_i\}$, $\{Z_i\}$, $\{Z_j\}$, and $\{Y_j\}$ for some i, j are input clauses. To resolve $\{X_w^*\}$, the unit clause $\{Z_w^*\}$ is necessary. If $\{Z_w\}$ is an input clause, the relation between X and Z is not refined by PUR. If $\{Z_w^*\}$ is not an input clause, $\{Y_w^*\}$, $\{Z_u^{*,h(\neg z)}\}$, and $\{Y_u^{*,h(\neg y)}\}$, for some u, are necessary. To resolve $\{Z_u^{*,h(\neg z)}\}$ the clauses $\{Y_u^{*,h(\neg z)}\}$, $\{Z_v^{*,h(\neg z)}\}$ and $\{Y_v^{*,h(\neg y)}\}$, for some $v \neq u$, are necessary. Then $\{Z_w^*\}$ can also be resolved using $\{Z_v^{*,h(\neg z)}\}$ and $\{Y_v^{*,h(\neg y)}\}$, which contradicts the assumption that it is resolved using $\{Z_u^{*,h(\neg z)}\}$ and $\{Y_u^{*,h(\neg y)}\}$.

5. $R = \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{EQ}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$: Suppose this relation holds between the regions X and Y with the pseudo region Z, then clauses of the type $\{\neg X_w^{*,h(\neg x)}, \neg Z_w^{*,h(\neg z)}, \neg X_v^*, Z_v^*\}$, $\{\neg X_w^{*,h(\neg x)}, \neg Z_w^{*,h(\neg z)}, X_v^*, \neg Z_v^*\}$, $\{\neg Z_w^*, \neg Y_w^*, \neg Z_v^*, Y_v^*\}$, and $\{\neg Z_w^*, \neg Y_w^*, Z_v^*, \neg Y_v^*\}$ for all $w, v \in W_0$ as well as $\{X_i\}$ and $\{Z_i\}$ for some i are input clauses. To resolve $\{X_w^*\}$, the unit clauses $\{Z_w^*\}$, $\{X_u^{*,h(\neg x)}\}$, and $\{Z_u^{*,h(\neg z)}\}$ are necessary for some u. $\{Z_u^{*,h(\neg z)}\}$ can be obtained either with an extended R_{γ^*}-chain from Z to $\widetilde{Z}_u$ passing X or passing Y. If it passes X, the clauses $\{X_v^{*,h(\neg x)}\}$, and $\{Z_v^{*,h(\neg z)}\}$ are necessary for some $v \neq u$, which is a contradiction. If it passes Y, the clauses $\{Y_u^{*,h(\neg z)}\}$, $\{Y_v^*\}$ and $\{Z_v^*\}$ are necessary for some $v \neq u$. $\{Z_v^*\}$ can be obtained in two ways:

(a) If $\{Z_v\}$ is an input clause, $\{Y_v\}$ can be resolved using the clauses $\{Z_s^*\}$ and $\{Y_s^*\}$ for some s. Then $\{Z_u^{*,h(\neg z)}\}$ can also be resolved using $\{Z_s^*\}$ and $\{Y_s^*\}$, which contradicts the assumption that it is resolved using $\{Z_v^*\}$ and $\{Y_v^*\}$.

(b) If the chain to obtain $\{Z_v^*\}$ passes X, the clauses $\{Z_s^{*,h(\neg z)}\}$ and $\{X_s^{*,h(\neg x)}\}$ for some s are necessary, if it passes Y, the clauses $\{Z_s\}$

and $\{Y_s\}$ for some s are necessary. Both possibilities result in a contradiction to previous assumptions.

6. R is composed by intersection: The intersection of two relations is a refinement of both relations, so if one of the relations is refined to one of R_γ, this also holds for the intersection of the two relations. Some relations are refinements of a relation of R_γ but are not part of R_γ. These relations can be treated as if they were part of R_γ, because the specified properties of Proposition 7.9 also hold for them.

As R_γ is closed under composition, the proof is completed. □

Theorem 7.17 *The path-consistency method decides* $\mathsf{RSAT}(\widehat{\mathcal{H}}_8)$.

Proof: Let Θ be an inconsistent set of $\widehat{\mathcal{H}}_8$ formulas, $p'(\Theta)$ the equivalent propositional Horn formula and $c'(\Theta)$ the corresponding set of clauses. Since Θ is inconsistent, the empty clause results from $c'(\Theta)$ with PUR. The empty clause is resolved by $\{X^*_w\}$ and $\{\neg X^*_w\}$. The same proof as in Theorem 7.11 can be applied here, when it is based on Lemma 7.16 instead of Lemma 7.10. The only difference is that X might be a pseudo region. In this case $\{\neg X^*_w\}$ must be input clause, as it can not be resolved with PUR. For this there must either be a region Y with $R_{\neg\beta}(Y, X)$ or a region Z with $R_{\neg\gamma,\neg C}(X, Z)$. As there is no such region, X can not be a pseudo region. □

This theorem can be easily transfered to RCC-5.

Corollary 7.18 *The path-consistency method decides* $\mathsf{RSAT}(\widehat{\mathcal{H}}_5)$.

Another interesting question is whether the path-consistency method also decides the minimal-label problem $\mathsf{RMIN}(\widehat{\mathcal{H}}_8)$. As the following theorem shows this is not the case even for the set $\widehat{\mathcal{H}}_5$.

Theorem 7.19 *The path-consistency method is not sufficient for solving the minimal-label problem* $\mathsf{RMIN}(\widehat{\mathcal{H}}_5)$.

Proof: Figure 10 shows a constraint graph that is path-consistent but not minimal. The relation between A and D can be refined to PO but not to PP. □

8 Applicability of the Maximal Tractable Set $\widehat{\mathcal{H}}_8$

In this section we will discuss some practical advantages of the theoretical results obtained so far. One obvious advantage of the maximal tractable subset $\widehat{\mathcal{H}}_8$

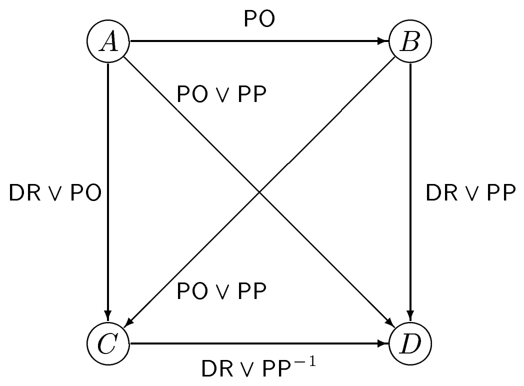

Figure 10: Path consistent but not minimal constraint graph.

is that the path-consistency method can now be used to decide RSAT when relations of $\widehat{\mathcal{H}}_8$ are used.

As in the case of temporal reasoning where the usage of the maximal tractable subset ORD-HORN has been extensively studied [18], $\widehat{\mathcal{H}}_8$ can also be used to speed up backtracking algorithms for the general NP-complete RSAT problem. Previously, every spatial formula had to be refined to a base relation before the path-consistency method could be applied to decide consistency. In the worst case this has to be done for all possible refinements. Supposing that the relations are uniformly distributed, the average branching factor, i.e. the average number of different refinements of a single relation to relations of $\mathcal{B}$ is 4.0.

Using our results it is sufficient to make refinements of all relations to relations of $\widehat{\mathcal{H}}_8$. Except for four relations, every relation not contained in $\widehat{\mathcal{H}}_8$ can be expressed as a union of two relations of $\widehat{\mathcal{H}}_8$, the four relations can only be expressed as a union of three $\widehat{\mathcal{H}}_8$ relations. This reduces the average branching factor to 1.4375. Both branching factors are of course worst-case measures because the search space can be considerably reduced when path-consistency is used as a forward checking method [12].

Table 3 shows an estimation of the worst-case running time for the average branching factors given above. All running times are computed as $b^{(n^2-n)/2}$ where b is the average branching factor and n the number of spatial variables contained in Θ. We assume that 100.000 path-consistency checks can be performed per second. This depends of course on the number of spatial variables, but it doesn't lead to big changes in the range we are considering. However, here we only intend to demonstrate the relative effect of using $\widehat{\mathcal{H}}_8$ instead of providing detailed empirical results.

$\|Reg(\Theta)\|$	$\mathcal{B}$ (4.0)	$\widehat{\mathcal{H}}_8$ (1.4375)
5	10sec	3msec
7	500days	20msec
10	10^{14}years	2min

Table 3: Comparison of the worst-case running times.

9 Summary

We analyzed the computational properties of the qualitative spatial calculus RCC-8 and identified the boundary between polynomial and NP-hard fragments. Using a modification of Bennett's encoding of RCC-8 in a multi-modal propositional logic, we transformed the RCC-8 consistency problem to a problem in propositional logic and isolated the relations that are representable as Horn clauses. As it turns out, the fragment identified in this way is also a maximal fragment that contains all base relations and is still computationally tractable. Further, we showed that for this fragment path-consistency is sufficient for deciding consistency. As in the case of qualitative temporal reasoning, our result allows to check whether the relations that are used in an application allow for a polynomial reasoning algorithm. Further, if the application requires an expressive power beyond the polynomial fragment, it can be used to speed up backtracking algorithms as in the case of qualitative temporal reasoning [17].

Research on this topic has to be continued, as it is still an open question whether there are other maximal tractable fragments of RCC-8 that also contain all base relations. Among other open problems, the question for a fragment that permits the determination of minimal labels by the path-consistency method seems to be interesting. Further, the determination of the computational properties of more expressive calculi like RCC-23 [2] and the design of efficient algorithms for the case of connected regions with a fixed dimensionality [11] appear to be interesting in an application context.

A Basics on Modal Logic

Propositional modal logic [8, 4] has the same syntax as standard propositional logic except for an additional unary operator $\Box$. One common approach to interpret modal logical formulas is the Kripke semantics, where models $\mathcal{M}$ are build upon so-called frames $\mathcal{F} = \langle W, R\rangle$ that consist of a set of worlds W together with an accessibility relation $R \subseteq W \times W$ defined on worlds, and a truth function π that assigns truth values to all the propositional atoms in every world. If a world $v \in W$ is accessible by a world $w \in W$, i.e. $(w, v) \in R$, we say that v is an R-successor of w and w an R-predecessor of v.

The truth of a modal formula ϕ in a world w of a model $\mathcal{M}$, written as $\mathcal{M}, w \Vdash \phi$, is defined on the inductive structure of ϕ:

$\mathcal{M}, w \Vdash a$ for an atom a	iff	$\pi(w, a) = \text{true}$
$\mathcal{M}, w \Vdash \neg\phi$	iff	$\mathcal{M}, w \nVdash \phi$
$\mathcal{M}, w \Vdash \phi \wedge \psi$	iff	$\mathcal{M}, w \Vdash \phi$ and $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \phi \vee \psi$	iff	$\mathcal{M}, w \Vdash \phi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \phi \rightarrow \psi$	iff	$\mathcal{M}, w \nVdash \phi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \Box\phi$	iff	for all u with wRu: $\mathcal{M}, u \Vdash \phi$

It can be seen that a modal operator is closely connected to the corresponding accessibility relation. A modal logic is determined by a set of frames, while frames can be determined either by specifying particular relations or axioms that must hold in all frames. Some well-known modal logics are given in the following table:

Name	Relation	Axiom
K	–	$\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
T	reflexive	$\Box\phi \rightarrow \phi$
4	transitive	$\Box\phi \rightarrow \Box\Box\phi$
5	euclidian	$\neg\Box\phi \rightarrow \Box\neg\Box\phi$
	(wRu and wRv implies uRv)	

K is the set of all frames. Other modal logics can be obtained by combining these logics. Two of them, which are of particular interest in this paper, are S4 and S5. S4 are all frames that are K and T as well as 4. S5-frames are both S4 and 5, they have the special property that worlds are clustered. All worlds of a particular cluster are mutually accessible. The modal operator is named according to the modal logic, so, e.g., the operator of an S5-frame is called S5 operator.

In a multi-modal logic it is possible to use more than one modal operator, where every operator $\Box_i$ has it's own accessibility relation R_i, defined on the same set of worlds. The $\Box$-rule defined above has to be changed to different $\Box_i$-rules: $\mathcal{M}, w \Vdash \Box_i\phi$ iff for all u with wR_iu: $\mathcal{M}, u \Vdash \phi$. The other rules stay the same.

B Enumeration of the Relations of $\widehat{\mathcal{H}}_8$

A complete list of all relations of $\widehat{\mathcal{H}}_8$ is given here. The list is separated into two parts. The first part contains all the relations of $\mathcal{H}_8$ together with their abbreviated form. The second part contains all relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$ and their successive composition of relations of $\mathcal{H}_8$. The composition of the relations is chosen according to Lemma 7.13. $\overline{R}$ specifies the complement of R, i.e. $\overline{R}$ contains all base relations not included in R.

B.1 Relations of $\mathcal{H}_8$

Relations	Abbreviations
$\{\mathsf{DC}\}$	α
$\{\mathsf{EC}\}$	$\neg\alpha \wedge A$
$\{\mathsf{DC}, \mathsf{EC}\}$	A
$\{\mathsf{PO}\}$	$\neg\beta \wedge \neg\gamma \wedge \neg A$
$\{\mathsf{DC}, \mathsf{PO}\}$	$\neg\beta \wedge \neg\gamma \wedge (\alpha \vee \neg A)$
$\{\mathsf{EC}, \mathsf{PO}\}$	$\neg\alpha \wedge \neg\beta \wedge \neg\gamma$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}\}$	$\neg\beta \wedge \neg\gamma$
$\{\mathsf{TPP}\}$	$\beta \wedge \neg\gamma \wedge \neg B$
$\{\mathsf{DC}, \mathsf{TPP}\}$	$\neg\gamma \wedge \neg B \wedge (\alpha \vee \beta)$
$\{\mathsf{EC}, \mathsf{TPP}\}$	$\neg\alpha \wedge \neg\gamma \wedge \neg B \wedge (A \vee \beta)$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{TPP}\}$	$\neg\gamma \wedge \neg B \wedge (A \vee \beta)$
$\{\mathsf{PO}, \mathsf{TPP}\}$	$\neg\gamma \wedge \neg A \wedge \neg B$
$\{\mathsf{DC}, \mathsf{PO}, \mathsf{TPP}\}$	$\neg\gamma \wedge \neg B \wedge (\alpha \vee \neg A)$
$\{\mathsf{EC}, \mathsf{PO}, \mathsf{TPP}\}$	$\neg\alpha \wedge \neg\gamma \wedge \neg B$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}\}$	$\neg\gamma \wedge \neg B$
$\{\mathsf{NTPP}\}$	$\neg\gamma \wedge B$
$\{\mathsf{DC}, \mathsf{NTPP}\}$	$\neg\gamma \wedge (\alpha \vee B)$
$\{\mathsf{EC}, \mathsf{NTPP}\}$	$\neg\alpha \wedge \neg\gamma \wedge (A \vee B)$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{NTPP}\}$	$\neg\gamma \wedge (A \vee B)$
$\{\mathsf{TPP}, \mathsf{NTPP}\}$	$\beta \wedge \neg\gamma$
$\{\mathsf{DC}, \mathsf{TPP}, \mathsf{NTPP}\}$	$\neg\gamma \wedge (\alpha \vee \beta)$
$\{\mathsf{EC}, \mathsf{TPP}, \mathsf{NTPP}\}$	$\neg\alpha \wedge \neg\gamma \wedge (A \vee \beta)$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{TPP}, \mathsf{NTPP}\}$	$\neg\gamma \wedge (A \vee \beta)$
$\{\mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}\}$	$\neg\gamma \wedge \neg A$
$\{\mathsf{DC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}\}$	$\neg\gamma \wedge (\alpha \vee \neg A)$
$\{\mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}\}$	$\neg\alpha \wedge \neg\gamma$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}\}$	$\neg\gamma$
$\{\mathsf{TPP}^{-1}\}$	$\neg\beta \wedge \gamma \wedge \neg C$
$\{\mathsf{DC}, \mathsf{TPP}^{-1}\}$	$\neg\beta \wedge \neg C \wedge (\alpha \vee \gamma)$
$\{\mathsf{EC}, \mathsf{TPP}^{-1}\}$	$\neg\alpha \wedge \neg\beta \wedge \neg C \wedge (A \vee \gamma)$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{TPP}^{-1}\}$	$\neg\beta \wedge \neg C \wedge (A \vee \gamma)$
$\{\mathsf{PO}, \mathsf{TPP}^{-1}\}$	$\neg\beta \wedge \neg A \wedge \neg C$
$\{\mathsf{DC}, \mathsf{PO}, \mathsf{TPP}^{-1}\}$	$\neg\beta \wedge \neg C \wedge (\alpha \vee \neg A)$
$\{\mathsf{EC}, \mathsf{PO}, \mathsf{TPP}^{-1}\}$	$\neg\alpha \wedge \neg\beta \wedge \neg C$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}^{-1}\}$	$\neg\beta \wedge \neg C$
$\{\mathsf{NTPP}^{-1}\}$	$\neg\beta \wedge C$
$\{\mathsf{DC}, \mathsf{NTPP}^{-1}\}$	$\neg\beta \wedge (\alpha \vee C)$
$\{\mathsf{EC}, \mathsf{NTPP}^{-1}\}$	$\neg\alpha \wedge \neg\beta \wedge (A \vee C)$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{NTPP}^{-1}\}$	$\neg\beta \wedge (A \vee C)$
$\{\mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$	$\neg\beta \wedge \gamma$
$\{\mathsf{DC}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$	$\neg\beta \wedge (\alpha \vee \gamma)$
$\{\mathsf{EC}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$	$\neg\alpha \wedge \neg\beta \wedge (A \vee \gamma)$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$	$\neg\beta \wedge (A \vee \gamma)$

Relations	Abbreviations
$\{\mathsf{PO}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$	$\neg\beta \wedge \neg A$
$\{\mathsf{DC}, \mathsf{PO}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$	$\neg\beta \wedge (\alpha \vee \neg A)$
$\{\mathsf{EC}, \mathsf{PO}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$	$\neg\alpha \wedge \neg\beta$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}$	$\neg\beta$
$\{\mathsf{EQ}\}$	$\beta \wedge \gamma$
$\{\mathsf{DC}, \mathsf{EQ}\}$	$(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
$\{\mathsf{EC}, \mathsf{EQ}\}$	$\neg\alpha \wedge (A \vee \beta) \wedge (A \vee \gamma)$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{EQ}\}$	$(A \vee \beta) \wedge (A \vee \gamma)$
$\{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\}$	β
$\{\mathsf{DC}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\}$	$(\alpha \vee \beta)$
$\{\mathsf{EC}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\}$	$\neg\alpha \wedge (A \vee \beta)$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\}$	$(A \vee \beta)$
$\{\mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}, \mathsf{EQ}\}$	γ
$\{\mathsf{DC}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}, \mathsf{EQ}\}$	$(\alpha \vee \gamma)$
$\{\mathsf{EC}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}, \mathsf{EQ}\}$	$\neg\alpha \wedge (A \vee \gamma)$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}, \mathsf{EQ}\}$	$(A \vee \gamma)$
$\{\mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}, \mathsf{EQ}\}$	$\neg A$
$\{\mathsf{DC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}, \mathsf{EQ}\}$	$(\alpha \vee \neg A)$
$\{\mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}, \mathsf{EQ}\}$	$\neg\alpha$

B.2 Composition of the relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$

Relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$	Composition
$\overline{\{\mathsf{EQ}\}}$	$\{\mathsf{EC}\} \circ \{\mathsf{DC}, \mathsf{PO}\}$
$\{\mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\}$	$\{\mathsf{EC}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\} \circ \{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\}$
$\overline{\{\mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}\}}$	$\{\mathsf{DC}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\} \circ \{\mathsf{TPP}, \mathsf{NTPP}, \mathsf{EQ}\}$
$\overline{\{\mathsf{NTPP}, \mathsf{NTPP}^{-1}\}}$	$\{\mathsf{EC}\} \circ \{\mathsf{EC}\}$
$\overline{\{\mathsf{DC}, \mathsf{EC}, \mathsf{NTPP}^{-1}\}}$	$\{\mathsf{PO}, \mathsf{TPP}^{-1}\} \circ \{\mathsf{TPP}\}$
$\overline{\{\mathsf{DC}, \mathsf{NTPP}^{-1}\}}$	$\{\mathsf{EC}, \mathsf{PO}, \mathsf{TPP}^{-1}\} \circ \{\mathsf{TPP}\}$
$\overline{\{\mathsf{NTPP}^{-1}\}}$	$\{\mathsf{EC}\} \circ \{\mathsf{EC}, \mathsf{PO}\}$
$\overline{\{\mathsf{NTPP}\}}$	$\{\mathsf{EC}\} \circ \{\mathsf{DC}, \mathsf{EC}\}$
$\{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{TPP}^{-1}\}$	$\overline{\{\mathsf{EQ}\}} \cap \overline{\{\mathsf{NTPP}, \mathsf{NTPP}^{-1}\}}$
$\{\mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}, \mathsf{TPP}^{-1}\}$	$\overline{\{\mathsf{EQ}\}} \cap \overline{\{\mathsf{DC}, \mathsf{EC}, \mathsf{NTPP}^{-1}\}}$
$\overline{\{\mathsf{DC}, \mathsf{NTPP}^{-1}, \mathsf{EQ}\}}$	$\overline{\{\mathsf{EQ}\}} \cap \overline{\{\mathsf{DC}, \mathsf{NTPP}^{-1}\}}$
$\overline{\{\mathsf{NTPP}^{-1}, \mathsf{EQ}\}}$	$\overline{\{\mathsf{EQ}\}} \cap \overline{\{\mathsf{NTPP}^{-1}\}}$
$\overline{\{\mathsf{NTPP}, \mathsf{EQ}\}}$	$\overline{\{\mathsf{EQ}\}} \cap \overline{\{\mathsf{NTPP}\}}$
$\overline{\{\mathsf{DC}, \mathsf{EC}, \mathsf{EQ}\}}$	$\overline{\{\mathsf{EQ}\}} \cap \overline{\{\mathsf{DC}, \mathsf{EC}\}}$
$\overline{\{\mathsf{EC}, \mathsf{EQ}\}}$	$\overline{\{\mathsf{EQ}\}} \cap \overline{\{\mathsf{EC}\}}$
$\overline{\{\mathsf{DC}, \mathsf{EQ}\}}$	$\overline{\{\mathsf{EQ}\}} \cap \overline{\{\mathsf{DC}\}}$

Relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$	Composition
$\{\mathsf{TPP},\mathsf{EQ}\}$	$\{\mathsf{TPP},\mathsf{NTPP},\mathsf{EQ}\} \cap \overline{\{\mathsf{NTPP},\mathsf{NTPP}^{-1}\}}$
$\{\mathsf{DC},\mathsf{TPP},\mathsf{EQ}\}$	$\{\mathsf{DC},\mathsf{TPP},\mathsf{NTPP},\mathsf{EQ}\} \cap \overline{\{\mathsf{NTPP},\mathsf{NTPP}^{-1}\}}$
$\{\mathsf{EC},\mathsf{TPP},\mathsf{EQ}\}$	$\{\mathsf{EC},\mathsf{TPP},\mathsf{NTPP},\mathsf{EQ}\} \cap \overline{\{\mathsf{NTPP},\mathsf{NTPP}^{-1}\}}$
$\{\mathsf{DC},\mathsf{EC},\mathsf{TPP},\mathsf{EQ}\}$	$\{\mathsf{DC},\mathsf{EC},\mathsf{TPP},\mathsf{NTPP},\mathsf{EQ}\} \cap \overline{\{\mathsf{NTPP},\mathsf{NTPP}^{-1}\}}$
$\{\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{EQ}\}$	$\{\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{NTPP},\mathsf{EQ}\} \cap \overline{\{\mathsf{NTPP},\mathsf{NTPP}^{-1}\}}$
$\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{EQ}\}$	$\overline{\{\mathsf{TPP}^{-1},\mathsf{NTPP}^{-1}\}} \cap \overline{\{\mathsf{NTPP},\mathsf{NTPP}^{-1}\}}$
$\{\mathsf{PO},\mathsf{TPP},\mathsf{NTPP},\mathsf{EQ}\}$	$\{\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{NTPP},\mathsf{EQ}\} \cap \overline{\{\mathsf{DC},\mathsf{EC},\mathsf{NTPP}^{-1}\}}$
$\{\mathsf{DC},\mathsf{PO},\mathsf{TPP},\mathsf{NTPP},\mathsf{EQ}\}$	$\overline{\{\mathsf{TPP}^{-1},\mathsf{NTPP}^{-1}\}} \cap \overline{\{\mathsf{EC}\}}$
$\{\mathsf{PO},\mathsf{TPP},\mathsf{TPP}^{-1},\mathsf{EQ}\}$	$\overline{\{\mathsf{NTPP},\mathsf{NTPP}^{-1}\}} \cap \overline{\{\mathsf{DC},\mathsf{EC},\mathsf{NTPP}^{-1}\}}$
$\{\mathsf{DC},\mathsf{PO},\mathsf{TPP},\mathsf{TPP}^{-1},\mathsf{EQ}\}$	$\overline{\{\mathsf{NTPP},\mathsf{NTPP}^{-1}\}} \cap \overline{\{\mathsf{EC}\}}$
$\{\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{TPP}^{-1},\mathsf{EQ}\}$	$\overline{\{\mathsf{NTPP},\mathsf{NTPP}^{-1}\}} \cap \overline{\{\mathsf{DC},\mathsf{NTPP}^{-1}\}}$
$\overline{\{\mathsf{EC},\mathsf{NTPP}^{-1}\}}$	$\overline{\{\mathsf{NTPP}^{-1}\}} \cap \overline{\{\mathsf{EC}\}}$
$\overline{\{\mathsf{DC},\mathsf{EC},\mathsf{NTPP}\}}$	$\overline{\{\mathsf{NTPP}\}} \cap \overline{\{\mathsf{DC},\mathsf{EC}\}}$
$\overline{\{\mathsf{EC},\mathsf{NTPP}\}}$	$\overline{\{\mathsf{NTPP}\}} \cap \overline{\{\mathsf{EC}\}}$
$\overline{\{\mathsf{DC},\mathsf{NTPP}\}}$	$\overline{\{\mathsf{NTPP}\}} \cap \overline{\{\mathsf{DC}\}}$
$\{\mathsf{DC},\mathsf{PO},\mathsf{TPP},\mathsf{TPP}^{-1}\}$	$\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{TPP}^{-1}\} \cap \overline{\{\mathsf{EC},\mathsf{EQ}\}}$
$\{\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{TPP}^{-1}\}$	$\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{TPP}^{-1}\} \cap \overline{\{\mathsf{DC},\mathsf{NTPP}^{-1},\mathsf{EQ}\}}$
$\{\mathsf{PO},\mathsf{TPP},\mathsf{TPP}^{-1}\}$	$\{\mathsf{DC},\mathsf{PO},\mathsf{TPP},\mathsf{TPP}^{-1}\} \cap \{\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{TPP}^{-1}\}$
$\overline{\{\mathsf{EC},\mathsf{NTPP}^{-1},\mathsf{EQ}\}}$	$\overline{\{\mathsf{NTPP}^{-1},\mathsf{EQ}\}} \cap \overline{\{\mathsf{EC},\mathsf{EQ}\}}$
$\overline{\{\mathsf{DC},\mathsf{EC},\mathsf{NTPP},\mathsf{EQ}\}}$	$\overline{\{\mathsf{NTPP},\mathsf{EQ}\}} \cap \overline{\{\mathsf{DC},\mathsf{EC},\mathsf{EQ}\}}$
$\overline{\{\mathsf{EC},\mathsf{NTPP},\mathsf{EQ}\}}$	$\overline{\{\mathsf{NTPP},\mathsf{EQ}\}} \cap \overline{\{\mathsf{EC},\mathsf{EQ}\}}$
$\overline{\{\mathsf{DC},\mathsf{NTPP},\mathsf{EQ}\}}$	$\overline{\{\mathsf{NTPP},\mathsf{EQ}\}} \cap \overline{\{\mathsf{DC},\mathsf{EQ}\}}$
$\{\mathsf{PO},\mathsf{TPP},\mathsf{EQ}\}$	$\{\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{EQ}\} \cap \{\mathsf{PO},\mathsf{TPP},\mathsf{NTPP},\mathsf{EQ}\}$
$\{\mathsf{DC},\mathsf{PO},\mathsf{TPP},\mathsf{EQ}\}$	$\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{EQ}\} \cap \{\mathsf{DC},\mathsf{PO},\mathsf{TPP},\mathsf{NTPP},\mathsf{EQ}\}$
$\overline{\{\mathsf{TPP},\mathsf{EQ}\}}$	$\{\mathsf{EC},\mathsf{NTPP}\} \circ \{\mathsf{DC},\mathsf{EQ}\}$
$\overline{\{\mathsf{TPP}^{-1},\mathsf{EQ}\}}$	$\{\mathsf{DC},\mathsf{EQ}\} \circ \{\mathsf{EC},\mathsf{NTPP}^{-1}\}$
$\overline{\{\mathsf{DC},\mathsf{TPP},\mathsf{NTPP}\}}$	$\{\mathsf{TPP}^{-1},\mathsf{NTPP}^{-1},\mathsf{EQ}\} \circ \{\mathsf{EC},\mathsf{EQ}\}$
$\overline{\{\mathsf{TPP},\mathsf{NTPP}\}}$	$\{\mathsf{EC},\mathsf{EQ}\} \circ \{\mathsf{DC},\mathsf{EQ}\}$
$\{\mathsf{PO},\mathsf{NTPP}\}$	$\{\mathsf{PO},\mathsf{TPP},\mathsf{NTPP}\} \cap \overline{\{\mathsf{TPP},\mathsf{EQ}\}}$
$\{\mathsf{DC},\mathsf{PO},\mathsf{NTPP}\}$	$\{\mathsf{DC},\mathsf{PO},\mathsf{TPP},\mathsf{NTPP}\} \cap \overline{\{\mathsf{TPP},\mathsf{EQ}\}}$
$\{\mathsf{EC},\mathsf{PO},\mathsf{NTPP}\}$	$\{\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{NTPP}\} \cap \overline{\{\mathsf{TPP},\mathsf{EQ}\}}$
$\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{NTPP}\}$	$\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{TPP},\mathsf{NTPP}\} \cap \overline{\{\mathsf{TPP},\mathsf{EQ}\}}$
$\{\mathsf{PO},\mathsf{NTPP},\mathsf{TPP}^{-1}\}$	$\{\mathsf{PO},\mathsf{TPP},\mathsf{NTPP},\mathsf{TPP}^{-1}\} \cap \overline{\{\mathsf{TPP},\mathsf{EQ}\}}$
$\{\mathsf{DC},\mathsf{PO},\mathsf{NTPP},\mathsf{TPP}^{-1}\}$	$\overline{\{\mathsf{EC},\mathsf{NTPP}^{-1},\mathsf{EQ}\}} \cap \overline{\{\mathsf{TPP},\mathsf{EQ}\}}$
$\{\mathsf{EC},\mathsf{PO},\mathsf{NTPP},\mathsf{TPP}^{-1}\}$	$\overline{\{\mathsf{DC},\mathsf{NTPP}^{-1},\mathsf{EQ}\}} \cap \overline{\{\mathsf{TPP},\mathsf{EQ}\}}$
$\{\mathsf{DC},\mathsf{EC},\mathsf{PO},\mathsf{NTPP},\mathsf{TPP}^{-1}\}$	$\overline{\{\mathsf{NTPP}^{-1},\mathsf{EQ}\}} \cap \overline{\{\mathsf{TPP},\mathsf{EQ}\}}$

Relations of $\widehat{\mathcal{H}}_8 \setminus \mathcal{H}_8$	Composition
$\{PO, NTPP^{-1}\}$	$\overline{\{TPP^{-1}, EQ\}} \cap \{PO, TPP^{-1}, NTPP^{-1}\}$
$\{DC, PO, NTPP^{-1}\}$	$\overline{\{TPP^{-1}, EQ\}} \cap \{DC, PO, TPP^{-1}, NTPP^{-1}\}$
$\{EC, PO, NTPP^{-1}\}$	$\overline{\{TPP^{-1}, EQ\}} \cap \{EC, PO, TPP^{-1}, NTPP^{-1}\}$
$\{DC, EC, PO, NTPP^{-1}\}$	$\overline{\{TPP^{-1}, EQ\}} \cap \{DC, EC, PO, TPP^{-1}, NTPP^{-1}\}$
$\{PO, TPP, NTPP^{-1}\}$	$\overline{\{TPP^{-1}, EQ\}} \cap \overline{\{DC, EC, NTPP, EQ\}}$
$\{DC, PO, TPP, NTPP^{-1}\}$	$\overline{\{TPP^{-1}, EQ\}} \cap \overline{\{EC, NTPP, EQ\}}$
$\{EC, PO, TPP, NTPP^{-1}\}$	$\overline{\{TPP^{-1}, EQ\}} \cap \overline{\{DC, NTPP, EQ\}}$
$\{DC, EC, PO, TPP, NTPP^{-1}\}$	$\overline{\{TPP^{-1}, EQ\}} \cap \overline{\{NTPP, EQ\}}$
$\overline{\{TPP, TPP^{-1}, EQ\}}$	$\overline{\{TPP^{-1}, EQ\}} \cap \overline{\{TPP, EQ\}}$
$\{PO, TPP, NTPP, NTPP^{-1}\}$	$\overline{\{TPP^{-1}, EQ\}} \cap \overline{\{DC, EC, EQ\}}$
$\overline{\{EC, TPP^{-1}, EQ\}}$	$\overline{\{TPP^{-1}, EQ\}} \cap \overline{\{EC, EQ\}}$
$\overline{\{DC, TPP^{-1}, EQ\}}$	$\overline{\{TPP^{-1}, EQ\}} \cap \overline{\{DC, EQ\}}$
$\overline{\{DC, EC, TPP, EQ\}}$	$\overline{\{TPP, EQ\}} \cap \overline{\{DC, EC, EQ\}}$
$\overline{\{EC, TPP, EQ\}}$	$\overline{\{TPP, EQ\}} \cap \overline{\{EC, EQ\}}$
$\overline{\{DC, TPP, EQ\}}$	$\overline{\{TPP, EQ\}} \cap \overline{\{DC, EQ\}}$
$\{PO, EQ\}$	$\{PO, TPP, EQ\} \cap \overline{\{DC, NTPP, TPP\}}$
$\{DC, PO, EQ\}$	$\{DC, PO, TPP, EQ\} \cap \overline{\{TPP, NTPP\}}$
$\{EC, PO, EQ\}$	$\{EC, PO, TPP, EQ\} \cap \overline{\{DC, NTPP, TPP\}}$
$\{DC, EC, PO, EQ\}$	$\{DC, EC, PO, TPP, EQ\} \cap \overline{\{TPP, NTPP\}}$
$\{TPP^{-1}, EQ\}$	$\{PO, TPP, TPP^{-1}, EQ\} \cap \overline{\{TPP^{-1}, NTPP^{-1}, EQ\}}$
$\{PO, TPP^{-1}, EQ\}$	$\{PO, TPP, TPP^{-1}, EQ\} \cap \overline{\{DC, NTPP, TPP\}}$
$\{DC, PO, TPP^{-1}, EQ\}$	$\{DC, PO, TPP, TPP^{-1}, EQ\} \cap \overline{\{TPP, NTPP\}}$
$\{EC, PO, TPP^{-1}, EQ\}$	$\{EC, PO, TPP, TPP^{-1}, EQ\} \cap \overline{\{DC, NTPP, TPP\}}$
$\{DC, EC, TPP^{-1}, EQ\}$	$\overline{\{NTPP, NTPP^{-1}\}} \cap \overline{\{PO, TPP, NTPP\}}$
$\{DC, TPP^{-1}, EQ\}$	$\{DC, EC, TPP^{-1}, EQ\} \cap \{DC, TPP^{-1}, NTPP^{-1}, EQ\}$
$\{EC, TPP^{-1}, EQ\}$	$\{DC, EC, TPP^{-1}, EQ\} \cap \{EC, TPP^{-1}, NTPP^{-1}, EQ\}$
$\{DC, EC, PO, TPP^{-1}, EQ\}$	$\overline{\{NTPP, NTPP^{-1}\}} \cap \overline{\{TPP, NTPP\}}$
$\{PO, TPP^{-1}, NTPP^{-1}, EQ\}$	$\overline{\{DC, NTPP, TPP\}} \cap \overline{\{DC, EC, NTPP\}}$
$\overline{\{EC, TPP, NTPP\}}$	$\overline{\{TPP, NTPP\}} \cap \overline{\{EC, NTPP\}}$
$\{PO, NTPP, NTPP^{-1}\}$	$\overline{\{TPP, TPP^{-1}, EQ\}} \cap \{PO, TPP, NTPP, NTPP^{-1}\}$
$\{DC, PO, NTPP, NTPP^{-1}\}$	$\overline{\{TPP, TPP^{-1}, EQ\}} \cap \overline{\{EC, TPP^{-1}, EQ\}}$
$\{EC, PO, NTPP, NTPP^{-1}\}$	$\overline{\{TPP, TPP^{-1}, EQ\}} \cap \overline{\{DC, TPP^{-1}, EQ\}}$

References

[1] James F. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843, November 1983.

[2] Brandon Bennett. Spatial reasoning with propositional logic. In J. Doyle, E. Sandewall, and P. Torasso, editors, *Principles of Knowledge Representation and Reasoning: Proceedings of the 4th International Conference*, pages 51–62, Bonn, Germany, May 1994. Morgan Kaufmann.

[3] Brandon Bennett. Modal logics for qualitative spatial reasoning. *Bulletin of the IGPL*, 4(1), 1995.

[4] B. F. Chellas. *Modal Logic: An Introduction.* Cambridge University Press, Cambridge, UK, 1980.

[5] William F. Dowling and Jean H. Gallier. Linear time algorithms for testing the satisfiability of propositional Horn formula. *The Journal of Logic Programming*, 3:267–284, 1984.

[6] Max J. Egenhofer. Reasoning about binary topological relations. In O. Günther and H.-J. Schek, editors, *Proceedings of the Second Symposium on Large Spatial Databases, SSD'91*, volume 525 of *Lecture Notes in Computer Science*, pages 143–160. Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[7] M. C. Fitting. *Proof Methods for Modal and Intuitionistic Logics.* Reidel, Dordrecht, Holland, 1983.

[8] M. C. Fitting. Basic modal logic. In D. M. Gabbay, C. J. Hogger, and J. A. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming – Vol. 1: Logical Foundations*, pages 365–448. Oxford, Clarendon Press, 1993.

[9] Michael R. Garey and David S. Johnson. *Computers and Intractability—A Guide to the Theory of NP-Completeness.* Freeman, San Francisco, CA, 1979.

[10] Martin C. Golumbic and Ron Shamir. Complexity and algorithms for reasoning about time: A graph-theoretic approach. *Journal of the Association for Computing Machinery*, 40(5):1128–1133, November 1993.

[11] Michelangelo Grigni, Dimitris Papadias, and Christos Papadimitriou. Topological inference. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence*, pages 901–906, Montreal, Canada, August 1995.

[12] Peter Ladkin and Alexander Reinefeld. Fast algebraic methods for interval constraint problems. *Annals of Mathematics and Artificial Intelligence*, to appear.

[13] Alan K. Mackworth. Consistency in networks of relations. *Artificial Intelligence*, 8:99–118, 1977.

[14] Alan K. Mackworth. Constraint satisfaction. In S. C. Shapiro, editor, *Encyclopedia of Artificial Intelligence*, pages 205–211. Wiley, Chichester, England, 1987.

[15] Alan K. Mackworth and Eugene C. Freuder. The complexity of some polynomial network consistency algorithms for constraint satisfaction problems. *Artificial Intelligence*, 25:65–73, 1985.

[16] Bernhard Nebel. Computational properties of qualitative spatial reasoning: First results. In I. Wachsmuth, C.-R. Rollinger, and W. Brauer, editors, *KI-95: Advances in Artificial Intelligence*, pages 233–244, Bielefeld, Germany, 1995. Springer-Verlag.

[17] Bernhard Nebel. Solving hard qualitative temporal reasoning problems: Evaluating the efficiency of using the ORD-Horn class. In *Proceedings of the 12th European Conference on Artificial Intelligence*, Amsterdam, The Netherlands, 1996. Wiley.

[18] Bernhard Nebel and Hans-Jürgen Bürckert. Reasoning about temporal relations: A maximal tractable subclass of Allen's interval algebra. *Journal of the Association for Computing Machinery*, 42(1):43–66, January 1995.

[19] David A. Randell, Anthony G. Cohn, and Zhan Cui. Computing transitivity tables: A challenge for automated theorem provers. In *Proceedings of the 11th CADE*. Springer-Verlag, 1992.

[20] David A. Randell, Zhan Cui, and Anthony G. Cohn. A spatial logic based on regions and connection. In B. Nebel, W. Swartout, and C. Rich, editors, *Principles of Knowledge Representation and Reasoning: Proceedings of the 3rd International Conference*, pages 165–176, Cambridge, MA, October 1992. Morgan Kaufmann.