The Ultimate Strategy to Search on m Rays?*

Technischer Bericht 100

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Abstract

We consider the problem of searching on m current rays for a target of unknown location. If no upper bound on the distance to the target is known in advance, then the optimal competitive ratio is $1 + 2m^m/(m-1)^{m-1}$. We show that if an upper bound of D on the distance to the target is known in advance, then the competitive ratio of any search strategy is at least $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ which is again optimal—but in a stricter sense.

To show the optimality of our lower bound we construct a search strategy that achieves this ratio. Surprisingly, our strategy does not need to know an upper bound on the distance to the target in advance; it achieves a competitive ratio of $1+2m^m/(m-1)^{m-1}-O(1/\log^2 D)$ if the target is found at distance D.

1 Introduction

Searching for a target is an important and well studied problem in robotics. In many realistic situations the robot does not possess complete knowledge about its environment, for instance, the robot may not have a map of its surroundings, or the location of the target may be unknown [DI94, IK95, Kle92, LOS95, PY89].

Since the robot has to make decisions about the search based only on the part of its environment that it has explored before, the search of the robot can be viewed as an *on-line* problem. One way to judge the performance of an on-line search strategy is to compare the distance traveled by the robot to the length of the shortest path from its starting point s to the target t. The ratio of the distance traveled by the robot to the optimal distance from s to t over all possible locations of the target is called the *competitive ratio* of the search strategy [ST85].

We are interested in obtaining upper and lower bounds on the competitive ratio of searching on m concurrent rays. Here, a point robot is imagined to stand at the origin of m rays and one of the rays contains the target t whose distance to the origin is unknown. The robot can only detect t if it stands on top of it. It can be shown that an optimal strategy visits the rays in cyclic order and increases the step length each time by a factor of $m/(m-1)$ starting with a step length of 1 [BYCR93, Gal80]. The competitive ratio C_m achieved by this strategy is given

*This research is supported by the DFG-Project "Diskrete Probleme", No. Ot 64/8-1.

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by $1+2m^m/(m-1)^{m-1}$. If randomization is used, the optimal competitive ratio is given by the minimum of the function $1+2a^m/((a-1)\ln a)$, for $a>1$ [Gal80, KRT93, KMS94].

Searching on m rays has proven to be a very useful tool for searching in a number of classes of simple polygons, such as star-shaped polygons [LOS97], generalized streets [DI94, LOS96], HV-streets [DHS95], and θ-streets [DHS95, Hip94].

However, the proof of optimality for the above m-way ray searching strategy relies on the unboundedness of the rays, that is, on the fact that the target can be placed arbitrarily far away from the starting point of the rays [BYCR93, Gal80]. But, if we consider polygons and the robot is equipped with a range finder, then it is possible to obtain an upper bound D on the distance to the target. In this case it is implicitly assumed that the strategy for searching on m-rays remains optimal though no proof of this assumption has been presented yet [DHS95, DI94, LOS96].

In this paper we provide the first lower bound proof for searching on m bounded rays; more precisely, we investigate the question if the knowledge of an upper bound on the distance to the target provides an advantage to the robot.

Let C_m^D be the optimal competitive ratio to search on m rays where the distance to the target is at most D. As mentioned above it is assumed that C_m^D approaches C_m as D goes to infinity; yet, there is only a proof for the case $m=2$ by López-Ortiz who shows that $9-O(1/\log D)$ is a lower bound for the competitive ratio of searching on two rays [LO96]. Icking *et al.* investigate the inverse problem [IKL97]. They consider the maximal reach of a strategy to search on the line if the competitive ratio of the strategy is given. The *reach* of a strategy X is the maximum distance D such that a target placed at a distance D to the origin is still detected by a robot using X if the competitive ratio is given. Since the competitive ratio is given, it is easy to derive a recurrence equation for the optimal reach which can be shown to be continuous and strictly monotone in D [IKL97]. This in turn implies that C_2^D is strictly monotone in D and assumes all values in the interval $[3,9]$.

In this paper we prove that

$$1+2m^m/(m-1)^{m-1}-O\left(1/\log^2 D\right) \tag{1}$$

is a lower bound on C_m^D, for general m; this also improves López-Ortiz' bound for $m=2$. Moreover, we present a strategy that achieves a competitive ratio of the same form as Equation 1, albeit with a different constant factor in the "big-Oh" term. Here, D is the distance at which the target is discovered. Astonishingly, our strategy achieves this competitive ratio without knowing an upper bound on D in advance. These two results imply that knowing an upper bound on the distance in advance does not improve the competitive ratio significantly. Note that all previously known strategies have a competitive ratio of $1+2m^m/(m-1)^{m-1}-O(1/D)$ if the target is detected at distance D.

The paper is organized as follows. In the next section we present some definitions concerning searching on m rays. In Section 3 we show that an optimal strategy to search on m bounded rays is periodic and monotone. In Section 4 we first consider searching on two rays to introduce our approach to analysing the competitive ratio of an optimal strategy. In Section 5 we generalize our ideas of the case of searching on two rays to m rays. Finally, in Section 6 we present a strategy whose competitive ratio converges asymptotically as fast to $1+2m^m/(m-1)^{m-1}$ as the lower bound which we have shown before.

2 Definitions

Let X be a strategy to search on m rays. We model X as a sequence of positive real numbers, that is, $X=(x_0,x_1,x_2,\dots)$ with $x_k>0$, for all $0\le k<\infty$. We illustrate this for the case of a point robot searching on the real line, that is, $m=2$.

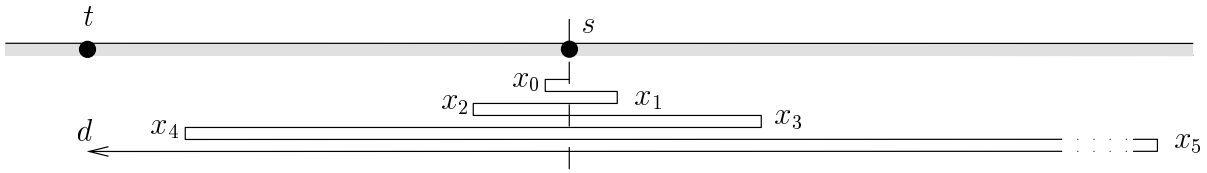

Figure 1: Searching on the real line.

In the beginning the position of the robot is a point s on the real line; it has to find a target t that is located somewhere to its left or right. It can only detect t if it stands on top of it. The robot starts at the origin s and travels for a distance of x_0 to one side, say to the left. The robot returns to s, travels a distance of x_1 to the right, returns, and so on. For illustration see Figure 1. Obviously, the values x_i which denote the distance that the robot travels to the left or to the right of s suffice to characterize a search strategy completely.

The Competitive Ratio

Assume that the target is discovered in Step $k+2$, say to the left of the origin. Clearly, the ray to the left of the origin was visited the last time before Step $k+2$ in Step k. Hence, the distance d to the target is greater than x_k. The distance traveled by the robot to discover t is $d + 2\sum_{i=0}^{k+1} x_i$. Since obviously $d > x_k$ and the target can be placed arbitrarily close to x_k by an adversary, the highest lower bound on the competitive ratio of Step k is given by the expression

$$\sup_{d>x_k} (d + 2\sum_{i=0}^{k+1} x_i)/d = \sup_{d>x_k} 1 + 2\sum_{i=0}^{k+1} x_i/d = 1 + 2\sum_{i=0}^{k+1} x_i/x_k.$$

Note that the above expression depends only on elements of X. For general m the competitive ratio is given by $1 + 2\sum_{i=0}^{k+m-1} x_i/x_k$ if the rays are visited cyclically.

The first step is a special case that we have not considered yet. If no information about the target is available, then one false move in the beginning may lead to an arbitrarily large competitive ratio. In order to avoid this problem we assume that a lower bound of one for the distance to the target t is known in advance.

3 Searching on m Bounded Rays

We are interested in the case that an upper bound D on the maximum distance of the target to the origin is known. The target may be placed on any of the m rays somewhere in the interval $[1, D]$, that is, we again assume that the lower bound on the distance to the target is one. We model a strategy X as a finite sequence of positive numbers, that is, $X = (x_0, \ldots, x_n)$, for some $n \geq 0$. The competitive ratio of X is denoted by C_X and the optimal competitive ratio of searching on m rays for a target that is placed at a distance of at most D from the origin is denoted by C_m^D. In the following we show that

$$C_m^D \geq 1 + 2\frac{m^m}{(m-1)^{m-1}} - O\left(\frac{1}{\log^2 D}\right).$$

3.1 Periodicity

In order to prove a lower bound on the competitive ratio, we first show that an optimal strategy, that is, a strategy with minimal competitive ratio is *periodic* and *monotone*. Let r_k be the ray

that the robot visits in Step k. A strategy is *periodic* if $r_{k+m} = r_k$, for all $0 \le k \le n-m$. A strategy is *monotone* if $x_{i+1} \ge x_i$, for all $0 \le i \le n-1$.

Lemma 3.1 *If X is a strategy to search on m rays for a target that is placed at a distance of at most D from the origin, then there is a monotone strategy X^* with $C_{X^*} \le C_X$.*

Proof: The proof is similar to the proof to the proof by Gal for the unbounded case [Gal80]. Let $X = (x_i)$ be a strategy to search m bounded rays and r_i the ray that is explored by X in the ith step. We define J_i as the index of the step in which ray r_i is visited the next time, that is, $J_i = \min\{j > i \mid r_j = r_i\}$. If there is no $j > i$ with $r_j = r_i$, then we define $J_i = i$. We represent X by the sequence of pairs (x_i, J_i).

Let $F_i(X) = \sum_{j=0}^{J_i-1} x_j/x_i$, for $0 \le i \le n$. If J_i does not equal i, then the competitive ratio in Step i of strategy X is given by $1 + 2F_i(X)$ which can be seen as in the case of searching on two rays. If J_i equals i, that is, $x_i = D$ and Step i is the last step on ray r_i, then the competitive ratio in Step i of strategy X is bounded by

$$\frac{2\sum_{j=0}^{i-1} x_j + d}{d} \le 1 + 2\frac{\sum_{j=0}^{i-1} x_j}{x_{J_i^{-1}}} = 1 + 2F_{J_i^{-1}}(X)$$

where J_i^{-1} is the index of the last visit of ray r_i before i and $d > x_{J_i^{-1}}$ is the distance from the origin to the target. Let I be the set of indizes i with $J_i \ne i$. The competitive ratio C_X of X is now given by

$$C_X = \max_{i \in I} 1 + 2F_i(X).$$

If X is monotone, then there is nothting to show. So assume that there is a Step k, $0 \le k \le n-1$ such that $x_{k+1} < x_k$. Let X^* be the search strategy which is equal to X except that for all steps $i \ge k$ the role of r_k and r_{k+1} is exchanged as are x_k and x_{k+1}. This can be achieved by setting $(x_k^*, J_k^*) = (x_{k+1}, J_{k+1})$ and $(x_{k+1}^*, J_{k+1}^*) = (x_k, J_k)$. For all other Steps i, $(x_i^*, J_i^*) = (x_i, J_i)$ unless $x_{k+1}^* = D$, in which case we set $J_{k+1}^* = k+1$ (and not equal to k as implied by the rule above). Note that $x_k^* = x_{k+1} = D$ is not possible since $x_{k+1} < x_k \le D$. Let I^* be the set of indizes i with $J_i^* \ne i$. We want to show that $C_{X^*} = \max_{i \in I^*} 1 + 2F_i(X^*) \le \max_{i \in I} 1 + 2F_i(X) = C_X$. Obviously, $F_i(X)$ and $F_i(X^*)$ differ only for the indices J_k^{-1}, J_{k+1}^{-1}, k, $k+1$ which we are going to consider more closely in the following.

First we assume that Step k is not the last step on ray r_k. (As mentioned before Step $k+1$ is never the last step on ray r_{k+1} as $x_{k+1} < x_k \le D$.)

$$\begin{aligned} F_k(X) &= \frac{\sum_{i=0}^{J_k-1} x_i}{x_k} = \frac{\sum_{i=0}^{J_{k+1}^*-1} x_i^*}{x_{k+1}^*} = F_{k+1}(X^*) \qquad \text{and} \\ F_{k+1}(X) &= \frac{\sum_{i=0}^{J_{k+1}-1} x_i}{x_{k+1}} = \frac{\sum_{i=0}^{J_k^*-1} x_i^*}{x_k^*} = F_k(X^*). \end{aligned}$$

Here the equalities follow from the fact that $J_{k+1}^* = J_k \ge k+2$ and $J_k^* = J_{k+1} \ge k+2$, that is, the exchange of x_k and x_{k+1} does not play a role in the summation. Next we consider Steps J_{k+1}^{-1} and J_k^{-1}. Note that ${J_k^{-1}}^* = J_k^{-1}$ and ${J_{k+1}^{-1}}^* = J_{k+1}^{-1}$. Moreover, $J_{J_k^{-1}} - 1 = J^*_{{J_k^{-1}}^*} - 1 = k-1$; therefore, $F_{J_k^{-1}}(X) = F_{{J_k^{-1}}^*}(X^*)$. This leaves us with Step J_{k+1}^{-1}. We have

$$F_{J_{k+1}^{-1}}(X) = \frac{\sum_{i=0}^{k} x_i}{x_{J_{k+1}^{-1}}} \ge \frac{\sum_{i=0}^{k} x_i - x_k + x_{k+1}}{x_{J_{k+1}^{-1}}} = \frac{\sum_{i=0}^{k-1} x_i^* + x_k^*}{x^*_{J_{k+1}^{-1}}} = F_{{J_{k+1}^{-1}}^*}(X^*).$$

Now assume that Step k is the last step on ray r_k and $D = x_k > x_{k+1}$. Then, $F_{k+1}(X^*) \leq F_{J_{k+1}^{-1}{}^*}(X^*)$. As above we obtain $F_k(X^*) = F_{k+1}(X)$, $F_{J_k^{-1}}(X^*) = F_{J_k^{-1}}(X)$ and $F_{J_{k+1}^{-1}{}^*}(X^*) \leq F_{J_{k+1}^{-1}}(X)$. Hence, the competitive ratio of Strategy X^* is no more than the competitive ratio of strategy X.

By performing bubble-sort on strategy X we see that there is a monotone strategy that has a competitive ratio no more than X which proves the claim. □

By Lemma 3.1 it suffices to consider monotone strategies in the following. Note that if X is monotone, then the last m steps of X all have length D, that is, there is an optimal strategy with $x_{n-m+1} = \cdots x_n = D$ and the set of indizes i with $J_i \neq i$ equals $\{0, \ldots, n-m\}$.

Lemma 3.2 *If X is a strategy to search on m rays for a target that is placed at a distance of at most D from the origin, then there is a periodic strategy X^* with $C_{X^*} \leq C_X$.*

Proof: Let X be strategy to search on m bounded rays. By Lemma 3.1 we can assume that X is monotone. We follow the proof idea of Yin [Yin94]. Let X^* consist of the same sequence of numbers except that X^* is now considered a periodic strategy. We consider the competitive ratios C_k of X and C_k^* of X^* in Step k. It suffices to show that, for every $0 \leq k \leq n-m$, there is a $0 \leq j \leq n-m$ with $C_k^* \leq C_j$. As mentioned above we do not need to consider the indices $n-m+1 \leq k \leq n$. So consider

$$C_k^* = 1 + 2\frac{\sum_{i=0}^{k+m-1} x_i}{x_k},$$

for some $0 \leq k \leq n-m$. For each ray r_j, $1 \leq j \leq m$, let k_j be the first time X explores ray r_j after Step k. Since $x_j < D$, for all $0 \leq j \leq n-m$, k_j exists, for all $1 \leq j \leq n-m$. Note that there is one ray r_l such that $k_l \geq k+m$. If r_l is explored before Step k, then let $j_l \leq k$ be the index of the last exploration; otherwise let $j_l = -1$ and $x_{j_l} = 1$. In both cases $x_{j_l} \leq x_k$ since X is monotone and

$$C_k^* = 1 + 2\frac{\sum_{i=0}^{k+m-1} x_i}{x_k} \leq 1 + 2\frac{\sum_{i=0}^{k_l-1} x_i}{x_{j_l}} = C_{j_l},$$

which implies that the competitive ratio of X is at least as large as the competitive ratio of X^*. □

3.2 A Recurrence Equation

In the following we assume that X is an optimal periodic, monotone strategy. As mentioned before F_k simplifies in this case to

$$F_k(X) = \sum_{i=0}^{k+m-1} x_i/x_k,$$

for $k = 0, \ldots, n-m$ and $C_X = \max_{0 \leq i \leq n-m} 1 + 2F_i(X)$. We now show that the values x_i satisfy a recurrence equation. The following lemma was proven by Katsoupias and Papadimitriou for the case $m = 2$ and unbounded rays [KPY96].

Lemma 3.3 *If X^* is an optimal strategy, then $1 + 2F_k(X^*) = C_m^D$, for all $0 \leq k \leq n-m$.*

Proof: The proof is by contradiction. It is based on the observation that F_k is the only function which is decreasing in x_k and all other functions F_i with $i \geq k-m+1$ are increasing in x_k [KPY96]. So if there is an index k with $1+2F_k(X) < C_m^D$, then there is an $\varepsilon > 0$ and a $\delta > 0$ such that if x_k is decreased by ε, then $1+2F_k(X') \leq C_m^D - \delta$ if X' is the sequence where x_k is replaced by $x_k - \varepsilon$ and, in addition, $1+2F_i(X') \leq C_m^D - \delta$, for all $k-m+1 \leq i \neq k \leq n-m$.

Let X be a sequence with competitive ratio C_m^D and l_X the minimal index for X with $1+2F_k(X) < C_m^D$. Let X^* be a sequence with competitive ratio C_m^D such that $l^* = l_{X^*}$ is minimal among all such sequences. If $l^* \geq m-1$, then we can apply the above argument and obtain a sequence X' from X^* with $1+2F_k(X') < C_m^D - \min\{\varepsilon, \delta\}$, for all $l^*-m+1 \leq k \leq n-m$—in contradiction to the minimality of l^*. If $l^* < m-1$, then we can apply the above argument and obtain a sequence X' from X^* with $1+2F_k(X') < C_m^D - \delta$, for all $0 \leq k \leq n-m$—in contradiction to the minimality of C_m^D. Hence, there is no sequence X with competitive ratio C_m^D and an index k with $1+2F_k(X) < C_m^D$. □

In the following let $c_m^D = (C_m^D - 1)/2$. Lemma 3.3 implies that an optimal strategy X satisfies the following recurrence equation.

$$\frac{\sum_{i=0}^{k+m-1} x_i}{x_k} = c_m^D \quad \text{or} \quad \sum_{i=0}^{k+m-1} x_i = c_m^D x_k, \tag{2}$$

for $0 \leq k \leq n-m$. Moreover, the first time ray m is visited, the competitive ratio is given by

$$1 + 2\frac{\sum_{i=0}^{m-1} x_i}{d_{min}} \leq 1 + 2c_m^D$$

where $d_{min} = 1$ is the lower bound on the distance to the target. If $1+2\sum_{i=0}^{m-1} x_i/d_{min} < 1+2c_m^D$, then we can multiply all step lenghts x_i by a factor λ such equality is achieved in (3); since $F_k(\lambda X) = F_k(X)$ the competitve ratio of X does not increase. If we set $x_{-1} = 1$, then Equation 2 even holds for $k = -1$.

Equation 2 completely defines the sequence $X = (1, x_0, x_1, \ldots, x_n)$ if we are given the values $x_0, \ldots, x_{m-2}$ (which we do not know); however, we know the values of $x_{n-m+1}, \ldots, x_n$ since $x_i = D$ in the last m steps. Unfortunately, the value of x_n is irrelevant since x_n does not appear in Equation system (2)—the m-th boundary value is given by $x_{-1} = 1$.

In order to obtain m consecutive boundary values, we take the following approach. Consider the value x_{n-m}. We have

$$x_{n-m} = \frac{\sum_{i=0}^{n-1} x_i}{c_m^D} \quad \text{or} \quad (1 - 1/c_m^D)x_{n-m} \geq \frac{\sum_{i=n-m+1}^{n-1} x_i}{c_m^D} \geq \frac{m-1}{c_m^D} D$$

or

$$x_{n-m} \geq \frac{m-1}{c_m^D - 1} D \geq \frac{m/2}{em} D \geq \frac{D}{2e}.$$

Let $X = (x_0, x_1, ..., x_n)$ be a strategy that satisfies the recurrence equation and the initial conditions $x_i \geq D/(2e)$, for $n-m \leq i \leq n-1$. If we reduce the last m values to $D/2e$, then the new strategy does not fulfill Equation 2 anymore but only

$$\sum_{i=0}^{k+m-1} x_i \leq c_m^D x_k,$$

for all $-1 \leq k \leq n-m$.

As in the proof of Lemma 3.3 we can now show that there is an optimal strategy X^*, that is, a strategy with minimal competitive ratio $1+2c$, that satisfies $x_i = D/(2e)$, for $n-m \leq i \leq n-1$, as initial conditions and, for all $-1 \leq k \leq n-m$,

$$\sum_{i=0}^{k+m-1} x_i = cx_k, \tag{3}$$

with $c \leq c_m^D$. Hence, it suffices to show a lower bound on c instead of c_m^D. We are now given the *last* m values of Equation 3. In order to make use of this information we consider the sequence Y of the values of X in reverse order, that is, $y_i = x_{n-i-1}$, for $i = 0, \ldots, n$.

Lemma 3.4 *The values y_i satisfy the following recurrence equation*

$$y_{k+m} - y_{k+m-1} + \frac{1}{c} y_k = 0, \tag{4}$$

for $0 \leq k \leq n-m$.

Proof: In order to show Equation 4, we only need to prove that X satisfies the equation

$$\begin{aligned} x_{n-(k+m)-1} - x_{n-(k+m-1)-1} + \frac{1}{c} x_{n-k-1} &= 0, \qquad \text{or} \\ x_{k+m-1} - cx_k + cx_{k-1} &= 0, \end{aligned} \tag{5}$$

for all $0 \leq k \leq n-m$. By Equations 3 we have

$$\sum_{i=0}^{k+m-1} x_i = cx_k, \tag{6}$$

for $0 \leq k \leq n-m$. The same equation also holds for $k-1$. Hence,

$$\sum_{i=0}^{k+m-1} x_i = c_m^D x_k \qquad \text{and} \qquad \sum_{i=0}^{k+m-2} x_i = c_m^D x_{k-1}.$$

By subtracting the second equation from the first we obtain Equation 5, for $0 \leq k \leq n-m$ as claimed. □

The starting values of Y are $y_0 = \cdots = y_{m-1} = D/2e$. In the following let $Y_{c,D}$ be the infinite sequence that is given by Equation 4 and by $y_0 = y_1 = \cdots = y_{m-1} = D/2e$. Note that $Y_{c,D}$ will also contain negative elements if $c < m^m/(m-1)^{m-1}$. In order to prove a lower bound on the optimal competitive ratio C_m^D we show the following lemma.

Lemma 3.5 *If $c < m^m/(m-1)^{m-1} - O(1/\log^2 D)$, then there is an index $k \geq m$ for the sequence $Y_{c,D} = (y_0, y_1, \ldots)$ with $y_{k-m} > c^2$ and $y_k < 0$.*

Assuming we have shown Lemma 3.5, we can prove easily that the competitive ratio of any strategy to search on m rays in the interval $[1, D]$ is bounded from below by $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$.

Theorem 3.6 *If $c < m^m/(m-1)^{m-1} - O(1/\log^2 D)$, then there is no strategy X with a competitive ratio of $1+2c$ that searches on m rays for a target of distance at most D to the origin.*

Proof: The proof is by contradiction. Assume there is a strategy X with a competitive ratio of $1+2c$ that searches on m rays for a target of distance at most D to the origin. This implies that $c_m^D \leq c$.

Let X^* be an optimal strategy to search on m rays. By Lemma 3.2 and the above considerations we can assume that X^* is periodic and satisfies Equation 2. Moreover, we can assume that there is a strategy X' with an even smaller competitive ratio c' that satisfies Equation 3 for c' and $x_{n-m} = \cdots = x_{n-1} = D/2e$. We define the values y_i by $y_i = x'_{n-i-1}$, for $0 \leq i \leq n-1$, where n is the length of X^*. The values of y_i satisfy Equation 4 (for c'), $\sum_{i=n-m+1}^{n-1} y_i = c'$, and $\sum_{i=n-m}^{n-1} y_i = c' y_{n-1}$.

The sequence $(y_0, \ldots, y_{n-1})$ is a prefix of the infinite sequence $Y_{c',D}$. Since $c' \leq c < m^m/(m-1)^{m-1} - O(1/\log^2 D)$, Lemma 3.5 implies that there is an index k for $Y_{c',D}$ with $y_{k-m} \geq c^2$ and $y_k < 0$. Since $y_k < 0$, n is at most k. If $n < k$, then we observe from Equation 4 that $y_i \geq y_{i+1}$ and, hence, $\sum_{i=n-m+1}^{n-1} y_i$ includes an element y_l with $l \leq k-m$; this implies that $\sum_{i=n-m+1}^{n-1} y_i$ is at least $c'^2 > c'$—a contradiction.

So assume that $n = k$. Since $y_{n-1} \leq \sum_{i=n-m+1}^{n-1} y_i \leq c'$, we have $\sum_{i=n-m}^{n-1} y_i > y_{n-m} > c'^2 > c' y_{n-1}$ which is again a contradiction. □

Since we are only interested in the asymptotic behaviour of $m^m/(m-1)^{m-1} - c_m^D$ we neglect the division by $2e$ in the following and assume that we are given $y_0 = y_1 = \cdots = y_{m-1} = D$ as initial values for Equation 4. This will not influence the asymptotic behaviour of $m^m/(m-1)^{m-1} - c_m^D$.

3.3 The Characteristic Equation

In the following we are only concerned with proving Lemma 3.5. Equation 4 has the characteristic equation

$$\lambda^m - \lambda^{m-1} + \frac{1}{c} = 0 \qquad \text{or} \qquad c = \frac{1}{\lambda^{m-1}(1-\lambda)}. \tag{7}$$

We first note that since $\lambda^{m-1}(1-\lambda) < 0$, for $\lambda > 1$, there is no positive real root larger than one. On the other hand, if we set $\mu = 1/\lambda$, then $c = \mu^m/(\mu-1)$ and if there is a positive real root λ of Equation 7 with $\lambda < 1$, then $c \geq \inf_{\mu>1} \mu^m/(\mu-1) = m^m/(m-1)^{m-1}$ and we are done. Hence, we can assume in the following that there is no positive real root of Equation 7.

So we investigate the complex and negative roots of Equation 7 in more detail.

4 Solving the Recurrence Equation for $m = 2$

In order to illustrate our approach we present the case $m = 2$ in greater detail. We can assume that c is less than $m^m/(m-1)^{m-1} = 4$ in the following.

4.1 An Explicit Solution

For $m = 2$ Equation 7 reduces to

$$\lambda^2 - \lambda + 1/c = 0 \tag{8}$$

with the solutions

$$\lambda \;=\; \frac{1}{2}\left(1 + i\sqrt{\frac{4-c}{c}}\right) \qquad \text{and} \qquad \overline{\lambda} \;=\; \frac{1}{2}\left(1 - i\sqrt{\frac{4-c}{c}}\right).$$

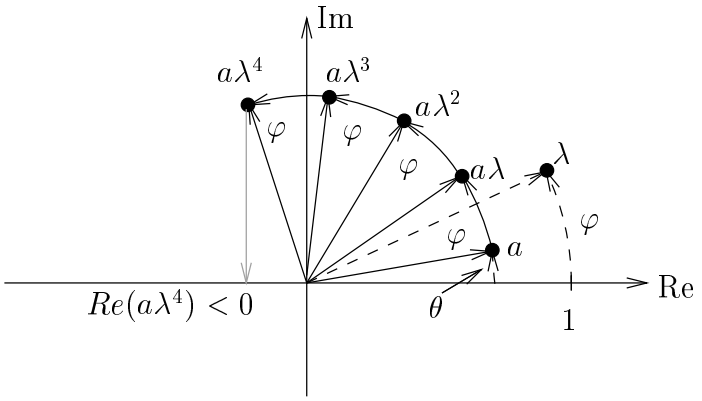

Figure 2: The sequence $2a\lambda^k$ turns by an angle of φ towards the second quadrant with each iteration.

Here, $\overline{\lambda}$ denotes the conjugate of λ. Hence, the solution of Equation 4 in the case $m = 2$ is given by

$$y_k = a\lambda^k + \overline{a}\overline{\lambda}^k = 2Re(a\lambda^k)$$

where Re denotes the real part of a complex number. Here, a and $\overline{a}$ are the solutions of the equation system

$$\begin{array}{rcrcl} a & + & \overline{a} & = & y_0 = D \\ a\lambda & + & \overline{a}\overline{\lambda} & = & y_1 = D \end{array}$$

which solves to

$$a = \frac{D}{2}\left(1 - i\sqrt{\frac{c}{4-c}}\right) \qquad \text{and} \qquad \overline{a} = \frac{D}{2}\left(1 + i\sqrt{\frac{c}{4-c}}\right)$$

4.2 Polar Coordinates

If we consider the polar-coordinates of λ and $\overline{\lambda}$, that is, $\lambda = \rho e^{i\varphi}$ and $\overline{\lambda} = \rho e^{i(-\varphi)}$, then $\rho = \sqrt{1/c}$ and $\varphi = \arctan(\sqrt{(4-c)/c})$. Similarly, for $a = \sigma e^{i\theta}$ and $\overline{a} = \sigma e^{i(-\theta)}$ we obtain $\sigma = D/\sqrt{4-c}$ and $\theta = -\arctan(\sqrt{c/(4-c)})$. Hence,

$$\begin{aligned} y_k = a\lambda^k + \overline{a}\overline{\lambda}^k &= \sigma\rho^k e^{i(k\varphi+\theta)} + \sigma\rho^k e^{-i(k\varphi+\theta)} \\ &= 2\sigma\rho^k \cos(k\varphi + \theta) \\ &= \frac{2D}{\sqrt{c^k(4-c)}} \cos\left(k \arctan\left(\sqrt{\frac{4-c}{c}}\right) - \arctan\left(\sqrt{\frac{c}{4-c}}\right)\right). \end{aligned}$$

If we visualize the above equation in the complex plane, then y_k is the projection of the vector of $2a\lambda^k$ onto the x-axis. If we multiply two complex numbers, then the radii are multiplied and the angles are added. Hence, the sequence $2a\lambda^k$ turns by an angle of φ towards the second quadrant with each iteration. Once $2a\lambda^k$ is in the second quadrant, $2Re(a\lambda^k)$ is negative. This is illustrated in Figure 2 (see also [Hip94, IKL97, Kle97]).

Hence, y_k becomes negative as soon as there is an integer $l \geq 0$ with

$$k \arctan\sqrt{(4-c)/c} - \arctan\sqrt{c/(4-c)} \in (\pi/2 + l\pi, 3\pi/2 + l\pi).$$

Note that since $\arctan(x) < \pi/2$, we can choose $l = 0$ in the above expression and there is a k with

$$k \arctan \sqrt{(4-c)/c} - \arctan \sqrt{c/(4-c)} \in (\pi/2, 3\pi/2).$$

We show that D can be chosen large enough such that there is an index n_0 with $y_{n_0} < 0$ and $y_{n_0-2} > c^2$ which proves Lemma 3.5. Of course, we are interested in the smallest D for which the above inequalities holds.

Let n_0 be the first index such that $y_{n_0} < 0$, that is,

$$\cos\left(n_0 \arctan\left(\sqrt{\frac{4-c}{c}}\right) - \arctan\left(\sqrt{\frac{c}{4-c}}\right)\right) < 0$$

or

$$n_0 = \left\lceil \frac{\arctan\left(\sqrt{\frac{c}{4-c}}\right) + \frac{\pi}{2}}{\arctan\left(\sqrt{\frac{4-c}{c}}\right)} \right\rceil.$$

Since n_0 is the smallest k such that $y_k < 0$,

$$(n_0 - 2)\arctan\left(\sqrt{\frac{4-c}{c}}\right) - \arctan\left(\sqrt{\frac{c}{4-c}}\right) \leq \frac{\pi}{2} - \arctan\left(\sqrt{\frac{4-c}{c}}\right). \tag{9}$$

W.l.o.g. we assume that y_{n_0} belongs to ray r_1. Since the search alternates between the two rays, the last point visited on ray r_1 has a distance of

$$\begin{aligned} y_{n_0-2} &= \frac{2D}{\sqrt{c^{n_0-2}(4-c)}} \cos\left((n_0-2)\arctan\left(\sqrt{\frac{4-c}{c}}\right) - \arctan\left(\sqrt{\frac{c}{4-c}}\right)\right) \\ &\overset{(9)}{\geq} \frac{2D}{\sqrt{c^{n_0-2}(4-c)}} \cos\left(\frac{\pi}{2} - \arctan\left(\sqrt{\frac{4-c}{c}}\right)\right) \end{aligned} \tag{10}$$

to the origin. Since

$$\begin{aligned} \cos\left(\frac{\pi}{2} - \arctan\left(\sqrt{\frac{4-c}{c}}\right)\right) &= \sin\left(\arctan\left(\sqrt{\frac{4-c}{c}}\right)\right) = \frac{\sqrt{(4-c)/c}}{\sqrt{1+(4-c)/c}} \\ &= \frac{\sqrt{c}}{2}\sqrt{\frac{4-c}{c}} = \frac{\sqrt{4-c}}{2}, \end{aligned} \tag{11}$$

we have

$$y_{n_0-2} \overset{(10,11)}{\geq} \frac{2D}{\sqrt{c^{n_0-2}(4-c)}} \frac{\sqrt{4-c}}{2} = \frac{D}{\sqrt{c^{n_0-2}}}$$

We first consider the case that $c \in [1, 3)$; then, $n_0 \leq \pi/\arctan(1/3) = 6$. Hence,

$$\begin{aligned} y_{n_0-2} &= \frac{2D}{\sqrt{c^{n_0-2}(4-c)}} \cos\left((n_0-2)\arctan\left(\sqrt{\frac{4-c}{c}}\right) - \arctan\left(\sqrt{\frac{c}{4-c}}\right)\right) \\ &\geq \frac{2D}{\sqrt{c^{n_0-2}(4-c)}} \frac{\sqrt{4-c}}{2} \geq \frac{D}{\sqrt{c^4}} \geq \frac{D}{9}. \end{aligned}$$

If $D > 81$, then $y_{n_0-2} > 9 > c^2$ and $y_{n_0} < 0$ which proves Lemma 3.5 for $m = 2$ and $c < 3$.

Now assume that $c \in [3, 4]$. Then, we have

$$n_0 = \left\lceil \frac{\arctan\left(\sqrt{\frac{c}{4-c}}\right) + \frac{\pi}{2}}{\arctan\left(\sqrt{\frac{4-c}{c}}\right)} \right\rceil \leq \frac{\pi/2 + \pi/2}{3/4\sqrt{(4-c)/c}} \leq \frac{4\pi}{3}\sqrt{\frac{c}{4-c}} \leq \frac{9}{\sqrt{4-c}}. \tag{12}$$

The first inequality stems from the fact that

1. $c \geq 3$, that is, $\sqrt{(4-c)/c} \leq 1/\sqrt{3}$ and
2. $\arctan(x)' = 1/(1+x^2)$, that is, $\arctan(x) \geq x/(1+x^2)$ since arcus tangens is concave on the positive axis. Hence, $\arctan(\sqrt{(4-c)/c}) \geq \sqrt{(4-c)/c}/(1+\sqrt{1/3}^2)$

We obtain for y_{n_0-2}

$$y_{n_0-2} \geq \frac{D}{\sqrt{c^{n_0-2}}} \overset{(12)}{\geq} \frac{D}{\sqrt{c^{9/\sqrt{4-c}}}}.$$

Proposition 4.1 *If $3 < c < 4 - 81/\log^2(D/16)$, then $D/\sqrt{c^{9/\sqrt{4-c}}} > c^2$.*

Proof: We have

$$\begin{array}{rcl} c < 4 - \frac{81}{\log^2(D/16)} & \Rightarrow & 4 - c > \frac{81}{\log^2(D/16)} \overset{(\log c<2)}{\Rightarrow} \\ \log D > \left(\frac{4.5}{\sqrt{4-c}} + 2\right)\log c & \Rightarrow & 2\log D > \left(\frac{9}{\sqrt{4-c}} + 4\right)\log c \Rightarrow \\ D^2 > c^{9/\sqrt{4-c}+4} & \Rightarrow & \frac{D}{\sqrt{c^{9/\sqrt{4-c}}}} > c^2. \end{array}$$

□

Let $3 < c < 4 - 81/\log^2(D/16)$. Proposition 4.1 implies that $y_{n_0-2} > c^2$ and $y_{n_0} < 0$ which proves Lemma 3.5.

5 Solving the Recurrence Equation for the General Case

We now return to the general case. As for the case $m = 2$ we want to show that if there are only complex or negative solutions to Equation 7, then the angle of the polar coordinates of the solutions turn towards a negative solution. However, the details are much more complicated than in the case $m = 2$ since we have many roots of Equation 7 and the solutions cannot be computed explicitly. One possibility to get around this problem is to use estimates on the angles and radii of the roots.

In the following we show that there is one root λ which has the largest radius among all roots of Equation 7. After a sufficiently large number of steps the contribution of λ dominates the contribution of all other roos. Once the contribution of λ becomes negative in Step k so does the step length y_k. This limit the number of steps Y can consist of and in turn D.

Let $\lambda_0, \ldots, \lambda_{m-1}$ be the roots of Equation 7. The solution of the recurrence is given by

$$y_k = a_0\lambda_0^k + a_0\lambda_0^k + \cdots + a_{m-1}\lambda_{m-1}^k.$$

We first investigate the structure of the roots λ_i, $0 \leq i \leq m-1$.

Let λ be a complex root of Equation 7. We consider the polar coordinates of λ, that is, we set $\lambda = \rho e^{i\varphi}$.

Lemma 5.1 *If $\lambda = \rho e^{i\varphi}$ is a complex root of Equation 7, then $\rho = \sin(m-1)\varphi/\sin m\varphi$ and $1/c = \rho^{m-1}(\sin\varphi/\sin m\varphi)$.*

Proof: Let $\lambda = \rho e^{i\varphi}$ be a complex root of Equation 7. We have $\lambda^{m-1} = \rho^{m-1} e^{i(m-1)\varphi}$ and

$$\begin{aligned}
\lambda^{m-1}(\lambda - 1) &= \rho^{m-1}\left(\cos(m-1)\varphi + i\sin(m-1)\varphi\right)\left(\rho\cos\varphi - 1 + i\rho\sin\varphi\right) \\
&= \rho^{m-1}\left(\cos(m-1)\varphi(\rho\cos\varphi - 1) - \sin(m-1)\varphi\rho\sin\varphi + \right. \\
&\quad \left. i(\cos(m-1)\varphi\rho\sin\varphi + \sin(m-1)\varphi(\rho\cos\varphi - 1))\right) \\
&= \rho^{m-1}\left(\cos(m-1)\varphi\rho\cos\varphi - \sin(m-1)\varphi\rho\sin\varphi - \cos(m-1)\varphi + \right. \\
&\quad \left. i(\cos(m-1)\varphi\rho\sin\varphi + \sin(m-1)\varphi\rho\cos\varphi - \sin(m-1)\varphi)\right) \\
&= \rho^{m-1}\left(\rho\cos m\varphi - \cos(m-1)\varphi + i(\rho\sin m\varphi - \sin(m-1)\varphi)\right).
\end{aligned}$$

Since $\lambda^{m-1}(\lambda - 1) = -1/c \in I\!R$, we obtain

$$\rho\sin m\varphi - \sin(m-1)\varphi = 0 \qquad \text{or} \qquad \rho = \frac{\sin(m-1)\varphi}{\sin m\varphi}. \tag{13}$$

The second claim follows from the equalities

$$\begin{aligned}
\frac{1}{c} &= -\lambda^{m-1}(\lambda - 1) \qquad \text{and} \\
\rho\cos m\varphi - \cos(m-1)\varphi &= \frac{\sin(m-1)\varphi}{\sin m\varphi}\cos m\varphi - \cos(m-1)\varphi \;=\; \frac{\sin\varphi}{\sin m\varphi}
\end{aligned} \tag{14}$$

□

Lemma 5.1 has the following consequence.

Corollary 5.2 *If $\lambda = \rho e^{i\varphi}$ is a complex root of Equation 7, then λ is solely determined by φ.*

5.1 The Polar Angle of a Root

We first concentrate on the polar angle of a root λ of Equation 7.

Lemma 5.3 *If $\lambda = \rho e^{i\varphi}$ is a complex root of Equation 7, then $\varphi \in [2k\pi/(m-1), (2k+1)\pi/m]$.*

Proof: Let $\lambda = \rho e^{i\varphi}$ be a complex root of Equation 7. Equation 13 implies that since $\rho > 0$ both $\sin m\varphi$ and $\sin(m-1)\varphi$ have the same sign, that is, $m\varphi$ and $(m-1)\varphi$ either both belong to $[2k\pi, (2k+1)\pi]$ or to $[(2k+1)\pi, (2k+2)\pi]$. Since $-1/c = \lambda^{m-1}(\lambda - 1) = -\rho^{m-1}\sin\varphi/\sin m\varphi < 0$ and $\rho > 0$ as well as $\sin\varphi \geq 0$ (since $0 \leq \varphi \leq \pi$, a second condition is $\sin m\varphi > 0$. If $\varphi \in ((2k+1)\pi/(m-1), (2k+2)\pi/m]$, then $\sin m\varphi < 0$. Hence, $\varphi \in [2k\pi/(m-1), (2k+1)\pi/m]$, for some $0 \leq k \leq \lfloor m/2 \rfloor$ as claimed. □

There is exactly one root λ_k for each interval $[2k\pi/(m-1), (2k+1)\pi/m]$ with $0 \leq k \leq \lfloor m/2 \rfloor$.

Lemma 5.4 *For $0 \leq k \leq \lfloor m/2 \rfloor$, there is exactly one root $\lambda_k = \rho_k e^{i\varphi_k}$ of Equation 7 with $\varphi_k \in [2k\pi/(m-1), (2k+1)\pi/m]$.*

Proof: Since λ is a continuous function of φ by Corollary 5.2, it suffices to show that $1/(\lambda^{m-1}(1-\lambda))$ is monotone in φ and that $1/(\lambda^{m-1}(1-\lambda))$ assumes a value less than and greater than c, for each interval $[2k\pi/(m-1), (2k+1)\pi/m]$ with $0 \leq k \leq \lfloor m/2 \rfloor$. We first show that $1/(\lambda^{m-1}(1-\lambda))$ is monotone in φ.

$$\begin{aligned}
\frac{\partial}{\partial\varphi}\frac{1}{\lambda^{m-1}(1-\lambda)} &= \frac{\partial}{\partial\varphi}\frac{\sin^m m\varphi \sin}{\sin^{m-1}(m-1)\varphi\sin\varphi} \\
&= \frac{\sin^{m-1} m\varphi}{\sin^m(m-1)\varphi}\frac{m^2\sin^2\varphi + \sin^2 m\varphi - 2m\sin\varphi\sin m\varphi\cos(m-1)\varphi}{\sin^2\varphi} \\
&\geq 0
\end{aligned}$$

since

$$\begin{aligned} & m^2 \sin^2\varphi + \sin^2 m\varphi - 2m \sin\varphi \sin m\varphi \cos(m-1)\varphi \\ \geq\ & m^2 \sin^2\varphi + \sin^2 m\varphi - 2m|\sin\varphi||\sin m\varphi| \\ =\ & (m|\sin\varphi| - |\sin m\varphi|)^2 \geq 0. \end{aligned}$$

Hence, there is at most one root of Equation 7 for every interval $[2k\pi/(m-1), (2k+1)\pi/m]$, for $0 \leq k \leq \lfloor m/2 \rfloor - 1$. Since $\sin\varphi/\sin m\varphi$ is continuous over $[2k\pi/(m-1), (2k+1)\pi/m]$ and its values range from ∞ to 0, there is also at least one root of of Equation 7 with a polar angle in $[2k\pi/(m-1), (2k+1)\pi/m]$, for $0 \leq k \leq \lfloor m/2 \rfloor - 1$. □

The above roots account for $\lfloor m/2 \rfloor$ roots of Equation 7. If m is odd, then there is one root $\lambda_{\lfloor m/2 \rfloor}$ with $\varphi_{\lfloor m/2 \rfloor} = 2\lfloor m/2 \rfloor \pi/(m-1) = (2\lfloor m/2 \rfloor + 1)\pi/m = \pi$, that is, $\lambda_{\lfloor m/2 \rfloor}$ is a negative real root. The remaining $\lfloor m/2 \rfloor$ roots are given by the conjugates $\overline{\lambda}_k = \rho_k e^{-i\varphi_k}$ of λ_k as in the case $m = 2$ since we have that if λ is a root of Equation 7, then

$$\begin{aligned} \overline{\lambda}^{m-1}(\overline{\lambda} - 1) &= \rho^{m-1} e^{-im\varphi}(\rho e^{-i\varphi} - 1) \\ &= \rho^{m-1}\,(\rho\cos(-m\varphi) - \cos(-(m-1)\varphi) \\ &\quad + i(\rho\sin(-m\varphi) - \sin(-(m-1)\varphi))) \\ &= \rho^{m-1}\,(\rho\cos m\varphi - \cos(m-1)\varphi + i(\sin(m-1)\varphi - \rho\sin m\varphi)) \\ &= \rho^{m-1}(\rho\cos m\varphi - \cos(m-1)\varphi) = -\frac{1}{c} \end{aligned}$$

since $\sin(m-1)\varphi - \rho\sin m\varphi = 0$. Hence, if λ is a root of Equation 7, then $\overline{\lambda}$ is also a root of Equation 7.

Let φ_k be the angle of the root in $[2k\pi/(m-1), (2k+1)\pi/m]$. In the following we calculate a lower bound on the size of φ_0 if $c < m^m/(m-1)^{m-1}$.

Lemma 5.5

$$\varphi_0 \geq \min\left\{ \frac{1}{m^{3/2}} \sqrt{\frac{m^m}{(m-1)^{m-1}} - c},\ \frac{1}{\sqrt{3}m} \right\}.$$

Proof: We assume that $\varphi_0 \in [0, \pi/\sqrt{3}m]$ since if $\varphi_0 \geq \pi/\sqrt{3}m$, then the claim trivially holds.

$$\begin{aligned} c = \frac{1}{\lambda_0^{m-1}(1-\lambda_0)} &= \left(\frac{\sin m\varphi_0}{\sin(m-1)\varphi_0}\right)^{m-1} \frac{\sin m\varphi_0}{\sin\varphi_0} \\ &\geq \left(1 + \frac{(m - m^3\varphi_0^2/6)\varphi_0}{(m-1)\varphi_0}\right)^{m-1} \frac{m\varphi_0 - (m\varphi_0)^3/6}{\varphi_0} \\ &\geq \left(1 + \frac{m - m^3\pi^2/(18m^2)}{(m-1)}\right)^{m-1} \frac{m\varphi_0 - (m\varphi_0)^3/6}{\varphi_0} \quad (\varphi_0 \leq \pi/\sqrt{3}m) \\ &\geq \left(\frac{m - 1 + m - m\pi^2/18}{m-1}\right)^{m-1} \left(m - \frac{m^3\varphi_0^2}{6}\right) \\ &\geq \left(\frac{m}{m-1}\right)^{m-1} \left(m - \frac{m^3\varphi_0^2}{6}\right) \\ &\geq \left(1 - \frac{m^2\varphi_0^2}{6}\right) \frac{m^m}{(m-1)^{m-1}}. \end{aligned}$$

Here we use that by the Taylor-expansion of $\sin x - x^3/6 \leq \sin(x) \leq x$ if $x \geq 0$. Since $m^m/(m-1)^{m-1} < em$, we have

$$\varphi_0 \geq \min\left\{\sqrt{\frac{6(m^m/(m-1)^{m-1}-c)}{em^3}}, \frac{1}{\sqrt{3}m}\right\} \geq \min\left\{\frac{1}{m^{3/2}}\sqrt{\frac{m^m}{(m-1)^{m-1}}-c}, \frac{1}{\sqrt{3}m}\right\} \quad (15)$$

as claimed. □

5.2 The Radius of a Root

We now consider the radius of a root of Equation 7. Let ρ_k be the radius of λ_k. In the following we show that $\rho_k \geq \rho_{k+1}$, for all $0 \leq k \leq \lceil m/2 \rceil - 1$.

Lemma 5.6 *For all* $0 \leq k \leq \lceil m/2 \rceil - 1$, $\rho_k \geq \rho_{k+1}$.

Proof: We first observe that

$$\lambda_k^{m-1}(1-\lambda_k) = |\lambda_k^{m-1}(1-\lambda_k)| = \rho_k^{m-1}\sqrt{\rho_k^2 - 2\rho_k\cos\varphi_k + 1}.$$

We show that $\rho_k^{m-1}\sqrt{\rho_k^2 - 2\rho_k\cos\varphi_k + 1}$ is monotonely increasing in ρ_k.

We consider the derivative of $\rho^{m-1}\sqrt{\rho^2 - 2\rho\cos\varphi + 1}$ with respect to ρ and obtain

$$\frac{\partial}{\partial\rho}\rho^{m-1}\sqrt{\rho^2 - 2\rho\cos(\varphi) + 1} \quad = \quad \frac{\rho^{m-2}(m\rho^2 - (2m-1)\rho\cos\varphi + (m-1))}{\sqrt{\rho^2 - 2\rho\cos\varphi + 1}}.$$

Hence, $\rho^{m-1}\sqrt{\rho^2 - 2\rho\cos\varphi + 1}$ has an extremum greater than zero with respect to ρ if

$$\rho^2 - \frac{2m-1}{m}\rho\cos\varphi + \frac{m-1}{m} \quad = \quad 0,$$

that is, if

$$\rho \quad = \quad \frac{1}{2}\frac{(2m-2)\cos\varphi \pm \sqrt{1-((2m-1)\sin\varphi)^2}}{m}. \quad (16)$$

The above equation implies that

$$\sin\varphi \leq \frac{1}{2m-1}$$

or $\varphi \leq \arcsin(1/(2m-1)) \leq 2/(2m-1) < 1/(m-1)$, for $m \geq 3$, as $\arcsin(x) \leq 2x$, for $0 \leq x \leq \pi/3$. Therefore, the root of Equation 16 is outside the interval $[2k\pi/(m-1), (2k+1)\pi/m]$ and $\rho_k^{m-1}\sqrt{\rho_k^2 - 2\rho_k\cos\varphi_k + 1}$ is monotonely increasing in ρ_k, for all $1 \leq k \leq \lceil m/2 \rceil$, but not for $k = 0$. We now show that this implies that $\rho_0 \geq \rho_1 \geq \cdots \geq \rho_{\lceil m/2 \rceil}$. Let $0 \leq k \leq \lceil m/2 \rceil - 1$. Since $\pi > \varphi_{k+1} > \varphi_k > 0$, we have, for $0 \leq k \leq \lceil m/2 \rceil - 1$,

$$\frac{1}{c} = \rho_k^{m-1}\sqrt{\rho_k^2 - 2\rho_k\cos\varphi_k + 1} < \rho_k^{m-1}\sqrt{\rho_k^2 - 2\rho_k\cos\varphi_{k+1} + 1}$$

and as $\rho_{k+1}^{m-1}\sqrt{\rho_{k+1}^2 - 2\rho_{k+1}\cos\varphi_{k+1} + 1}$ is monotone in ρ_{k+1}, ρ_{k+1} has to be decreased in order to obtain equality. □

In the following we investigate the ratio ρ_0/ρ_k.

Lemma 5.7 $\rho_0/\rho_k \geq 1 + 1/(4m^3)$, *for all* $1 \leq k \leq \lceil m/2 \rceil$.

Proof: Since by Lemma 5.6 $\rho_1 \geq \rho_k$, for all for all $2 \leq k \leq \lceil m/2 \rceil$, it suffices to show that $\rho_0/\rho_1 \geq 1 + 1/(4m^3)$. Let f be the function

$$f(\varphi, \rho) = \left|\lambda^{m-1}(1-\lambda)\right|^2 = \rho^{2(m-1)}(\rho^2 - 2\rho\cos\varphi + 1).$$

Note that $f(\varphi_0, \rho_0) = f(\varphi_1, \rho_1) = 1/c^2$ and, therefore,

$$f(\varphi_1, \rho_0) - f(\varphi_0, \rho_0) \quad = \quad f(\varphi_1, \rho_0) - f(\varphi_1, \rho_1).$$

Now

$$f(\varphi_1, \rho_0) - f(\varphi_0, \rho_0) \quad = \quad 2\rho_0^{2m-1}(\cos\varphi_0 - \cos\varphi_1)$$

and

$$f(\varphi_1, \rho_0) - f(\varphi_1, \rho_1) \quad = \quad \int_{\rho_1}^{\rho_0} \frac{\partial}{\partial\rho} f(\varphi_1, \rho) d\rho \leq (\rho_0 - \rho_1) \max_{\rho \in [\rho_1, \rho_0]} \frac{\partial}{\partial\rho} f(\varphi_1, \rho).$$

If we consider the derivative of f with respect to ρ, then

$$\begin{aligned} \frac{\partial}{\partial\rho} f(\varphi_1, \rho) &= 2m\rho^{2m-1} - 2(2m-1)\rho^{2(m-1)}\cos\varphi + 2(m-1)\rho^{2m-3} \\ &= 2m\rho^{2m-3}\left(\rho^2 - 2\frac{2m-1}{2m}\rho\cos\varphi_1 + \frac{2(m-1)}{2m}\right). \end{aligned}$$

Hence,

$$\begin{aligned} f(\varphi_1, \rho_0) - f(\varphi_1, \rho_1) &\leq (\rho_0 - \rho_1) \max_{\rho \in [\rho_1, \rho_0]} 2m\rho^{2m-3}\left(\rho^2 - \frac{2m-1}{m}\rho\cos\varphi_1 + \frac{2(m-1)}{2m}\right) \\ &\leq (\rho_0 - \rho_1) \max_{\rho \in [\rho_1, \rho_0]} 2m\rho^{2m-3}\left(\rho^2 + |\frac{2m-1}{m}\rho\cos\varphi_1| + \frac{2(m-1)}{2m}\right) \\ &\leq (\rho_0 - \rho_1) 2m\rho_0^{2m-3}(\rho_0 + 1)^2 \end{aligned}$$

and, thus,

$$\begin{aligned} 2\rho_0^{2m-1}(\cos\varphi_0 - \cos\varphi_1) &\leq (\rho_0 - \rho_1)2m\rho_0^{2m-3}(\rho_0+1)^2 \qquad \Longleftrightarrow \\ \frac{\rho_0}{\rho_1}\frac{\rho_0(\cos\varphi_0 - \cos\varphi_1)}{m(\rho_0+1)^2} &\leq \frac{\rho_0}{\rho_1} - 1 \end{aligned}$$

or

$$\frac{\rho_0}{\rho_1} \quad \geq \quad \frac{1}{1 - \rho_0(\cos\varphi_0 - \cos\varphi_1)/(m(\rho_0+1)^2)} \geq 1 + \frac{\rho_0(\cos\varphi_0 - \cos\varphi_1)}{m(\rho_0+1)^2}.$$

In order to bound $\rho_0(\cos\varphi_0 - \cos\varphi_1)/(m(\rho_0+1)^2)$ from below, we need upper and lower bounds for ρ_0. We first give an upper bound. Observe that

$$\begin{aligned} &\lambda^{m-1}(1-\lambda) = \left(\frac{\sin(m-1)\varphi_0}{\sin m\varphi_0}\right)^{m-1}\frac{\sin\varphi_0}{\sin m\varphi_0} = \left(\frac{\sin(m-1)\varphi_0}{\sin m\varphi_0}\right)^{m}\frac{\sin\varphi_0}{\sin(m-1)\varphi_0} = \frac{1}{c} \\ &\Rightarrow \rho_0^m = \left(\frac{\sin(m-1)\varphi_0}{\sin m\varphi_0}\right)^m = \frac{\sin(m-1)\varphi_0}{\sin\varphi_0 c} \leq \frac{m-1}{c}. \end{aligned}$$

Hence, $\rho_0 \leq \sqrt[m]{(m-1)/c} \leq 1$ since $c \geq 3$.

Now we bound ρ_0 from below. Note that $|1-\lambda_0|$ is the distance between the point $(1,0)$ and the point λ_0 in the complex plane. Since λ_0 belongs to the wedge S_0 of numbers whose polar angle is in $[0,\pi/3]$ and whose radius is less than one, it is easy to see that the origin is the furthest point in S_0 from $(1,0)$ and $|1-\lambda_0| \leq 1$. Hence, $\rho_0 \geq \sqrt[m-1]{1/(|1-\lambda_0|c)} \geq \sqrt[m-1]{1/c}$. Since we assume that $c < m^m/(m-1)^{m-1} < em$, we obtain, $\rho_0 \geq \sqrt[m-1]{1/(em)} \geq 1/3$.

Next we give a lower bound for $\cos\varphi_0 - \cos\varphi_1$. Since $\varphi_0 \in [0,\pi/m]$ and $\varphi_1 \in [2\pi/(m-1), 3\pi/m]$ both of which are contained in $[0,\pi]$, for $m \geq 3$, $\cos\varphi_0 - \cos\varphi_1 \geq \cos\pi/m - \cos 2\pi/(m-1)$. Moreover, since cosine is concave over $[0,\pi/2]$ and $2\pi/(m-1) \leq \pi/2$, for $m \geq 5$,

$$\cos\varphi_0 - \cos\varphi_1 \geq \cos\frac{\pi}{m} - \cos\frac{2\pi}{m-1} \geq \sin\frac{\pi}{m}\left(\frac{2\pi}{m-1} - \frac{\pi}{m}\right) \geq \frac{\pi}{2m}\frac{\pi}{m} \geq \frac{\pi^2}{2m^2},$$

for $m \geq 5$. On the other hand, if $m = 3$, then $\cos(\pi/3) - \cos(2\pi/2) > 1 > \pi^2/18$ and if $m = 4$, then $\cos(\pi/4) - \cos(2\pi/3) > 1/\sqrt{2} > \pi^2/32$, so that the inequality $\cos\varphi_0 - \cos\varphi_1 \geq \pi^2/(2m^2)$ holds for all $m \geq 3$.

Hence, for $1 \leq k \leq \lceil m/2 \rceil$,

$$\frac{\rho_0}{\rho_k} \geq \frac{\rho_0}{\rho_1} \quad \geq \quad 1 + \frac{\pi^2}{6m^3(1+1)^2} \geq 1 + \frac{1}{4m^3}.$$

□

5.3 The Coefficients

We finally give an upper bound on the radius of the coefficients. Recall that the solution of Recurrence Equation 7 is given by

$$y_k = a_0\lambda_0^k + a_1\lambda_1^k + \cdots + a_{m-1}\lambda_{m-1}^k.$$

The coefficients a_i are the solution of the linear equation system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{m-1} & \lambda_1^{m-1} & \cdots & \lambda_{m-1}^{m-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{pmatrix} = \begin{pmatrix} D \\ D \\ \vdots \\ D \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{m-1} & \lambda_1^{m-1} & \cdots & \lambda_{m-1}^{m-1} \end{pmatrix}$$

and $A_i(x)$ the matrix A where the ith column is replaced by the vector $(x,\ldots,x)^T$. By Cramer's rule a_i is given as

$$\begin{aligned} a_i &= \det(A_i(D))/\det(A) = D\det(A_i(1))/\det(A) \\ &= D\frac{\prod_{j=0,j\neq 0}^{m-1}(1-\lambda_j)}{\prod_{j=0,j\neq i}^{m-1}(\lambda_i - \lambda_j)} \end{aligned} \tag{17}$$

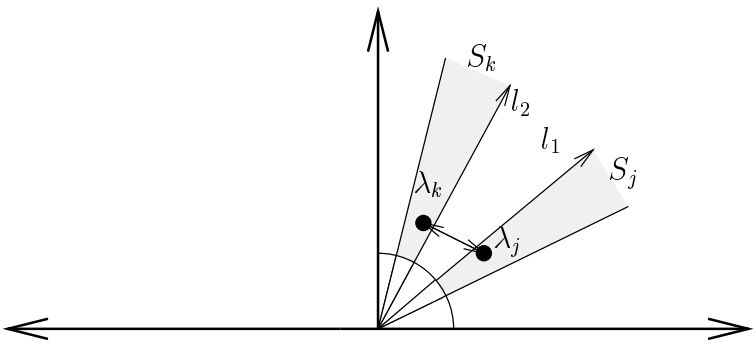

Figure 3: The sectors that λ_k and λ_j belong to.

since both A and $A_i(1)$ are Vandermonde matrices with

$$\begin{aligned} \det(A_i(1)) &= \prod_{j=0, j\neq i}^{m-1} (1-\lambda_j) \prod_{j<k, j,k\neq i} (\lambda_k - \lambda_j) \qquad \text{and} \\ \det(A) &= \prod_{j=0, j\neq i}^{m-1} (\lambda_i - \lambda_j) \prod_{j<k, j,k\neq i} (\lambda_k - \lambda_j). \end{aligned}$$

In order to bound the size of the ratio of $|a_i/a_0|$ we have the following lemma.

Lemma 5.8

$$\left|\frac{a_i}{a_0}\right| \leq 4^{2m} m^m.$$

Proof: We have

$$\begin{aligned} \left|\frac{a_i}{a_0}\right| &= \left|\frac{1-\lambda_i}{1-\lambda_0}\right| \left|\frac{\prod_{j=0, j\neq 0}^{m-1}(\lambda_0 - \lambda_j)}{\prod_{j=0, j\neq i}^{m-1}(\lambda_i - \lambda_j)}\right| \\ &\leq \frac{1+|\lambda_i|}{|1-\lambda_0|} \frac{\prod_{j=0, j\neq 0}^{m-1}(|\lambda_0| + |\lambda_j|)}{\prod_{j=0, j\neq i}^{m-1}|\lambda_i - \lambda_j|} \\ &\leq \frac{2}{|1-\lambda_0|} \frac{2^{m-1}}{\prod_{j=0, j\neq i}^{m-1}|\lambda_i - \lambda_j|} \end{aligned}$$

In order to obtain a lower bound for $|1-\lambda_0|$ we observe that

$$|1-\lambda_0| = \frac{1}{c|\lambda_0^{m-1}|} \geq \frac{1}{c} \geq \frac{1}{em} \tag{18}$$

Finally, we give a lower bound for $|\lambda_k - \lambda_j|$. Since $|1-\lambda_i| \leq 1+|\lambda_i| \leq 2$, $\lambda_i^{m-1} \geq 1/(2c) \geq 1/(2em)$ or $\lambda_i \geq \sqrt[m-1]{1/(2em)} \geq 1/5$.

If we view λ_k and λ_j as two points in the complex plane, then λ_k is contained in the angular sector of $S_k = [2k\pi/(m-1), (2k+1)\pi/m]$ and λ_j is contained in the angular sector of $S_j = [2j\pi/(m-1), (2j+1)\pi/m]$ (see Figure 3). Since $|\lambda_k| \geq 1/5$ and $|\lambda_j| \geq 1/5$, the distance between λ_k and λ_j is at least the distance between the points of S_k and S_j outside the circle with radius $1/5$. W.l.o.g. assume that $k > j$. Let l_1 be the line with angle $2k\pi/(m-1)$ through the origin and l_2 be the line with angle $(2j+1)\pi/m$ through the origin. If p is the point on l_1

with distance $1/5$ to the origin, then the distance of S_k to S_j outside the circle with radius $1/5$ is at most the distance of p to l_2. By elementary geometry we obtain that

$$|\lambda_k - \lambda_j| \geq d(p, l_2) \quad = \quad \frac{\sin\left(2k\pi/(m-1) - (2j+1)\pi/m\right)}{5} \geq \frac{\pi}{10m} \geq \frac{1}{4m}. \tag{19}$$

Combining the estimates for $|1-\lambda_0|$ and $|\lambda_k - \lambda_j|$ we obtain

$$\left|\frac{a_i}{a_0}\right| \quad \leq \quad \frac{2^m}{|1-\lambda_0| \prod_{j=0, j\neq i}^{m-1} |\lambda_i - \lambda_j|} \leq 2^m em (4m)^{m-1} \leq 4^{2m} m^m$$

as claimed. □

The following lemma gives a lower bound of the absolute value of a_0.

Lemma 5.9

$$|a_0| > \frac{D}{(2em)^{m-1}}.$$

Proof: The proof follows easily from Equations 17 and 18.

$$|a_0| = D \frac{\prod_{j=1}^{m-1} |1-\lambda_j|}{\prod_{j=1}^{m-1} |\lambda_0 - \lambda_j|} \geq D \frac{(1/em)^{m-1}}{2^{m-1}}.$$

Note that the lower bound for $|1-\lambda_0|$ of Equation 18 is also a lower bound for $|1-\lambda_i|$ and that $|\lambda_0 - \lambda_j| \leq \rho_0 + \rho_j < 2$. □

5.4 Putting it all Together

We now put the estimates we obtained for the radii and the angles of the roots of Equation 7 as well as the coefficients into use. W.l.o.g. we assume that m is even. If m is odd, then an analogous proof works. We start off by proving a lower and an upper bound on the size of y_k.

Lemma 5.10

$$y_k \quad \leq \quad 2|a_0|\rho_0^k \left(\cos(\theta_0 + k\varphi_0) + \frac{4^{2m} m^{m+1}}{(1 + 1/(4m^3))^k} \right).$$

and

$$y_k \quad \geq \quad 2|a_0|\rho_0^k \left(\cos(\theta_0 + k\varphi_0) - \frac{4^{2m} m^{m+1}}{(1 + 1/(4m^3))^k} \right).$$

Proof: Recall that

$$y_k \quad = \quad \sum_{j=0}^{\lfloor m/2 \rfloor} a_j \lambda_j^k + \overline{a}_j \overline{\lambda}_j^k \leq a_0 \lambda_0^k + \overline{a}_0 \overline{\lambda}_0^k + \sum_{j=0}^{\lfloor m/2 \rfloor} 2|a_j \lambda_j^k|.$$

If $\lambda_0 = \rho_0 e^{i\varphi_0}$ and $a_0 = \sigma_0 e^{i\theta_0}$, then

$$a_0 \lambda_0^k + \overline{a}_0 \overline{\lambda}_0^k \quad = \quad \sigma_0 \rho_0^k e^{i(\theta_0 + k\varphi_0)} + \sigma_0 \rho_0^k e^{-i(\theta_0 + k\varphi_0)} = 2\sigma_0 \rho_0^k \cos(\theta_0 + k\varphi_0).$$

and

$$\begin{aligned} y_k &\leq 2|a_0|\rho_0^k \left(\cos(\theta_0 + k\varphi_0) + \sum_{j=0}^{\lfloor m/2 \rfloor} \left|\frac{a_j}{a_0}\right| \left|\frac{\rho_j^k}{\rho_0^k}\right| \right) \\ &\leq 2|a_0|\rho_0^k \left(\cos(\theta_0 + k\varphi_0) + \frac{4^{2m} m^{m+1}}{(1 + 1/(4m^3))^k} \right) \end{aligned}$$

by Lemmas 5.7 and 5.8. Similarly,

$$\begin{aligned} y_k &\geq 2|a_0|\rho_0^k \left(\cos(\theta_0 + k\varphi_0) - \sum_{j=0}^{\lfloor m/2 \rfloor} \left|\frac{a_j}{a_0}\right| \left|\frac{\rho_j^k}{\rho_0^k}\right| \right) \\ &\geq 2|a_0|\rho_0^k \left(\cos(\theta_0 + k\varphi_0) - \frac{4^{2m} m^{m+1}}{(1 + 1/(4m^3))^k} \right). \end{aligned}$$

□

In the following let $\varepsilon = \sqrt{m^m/(m-1)^{m-1} - c}$. We assume that $\varepsilon < 1$. The case $\varepsilon \geq 1$ can be treated as the case $c \leq 3$ in the case $m = 2$. Let k_0 be the first index greater than $4m^3(3m \log m - \log \varepsilon) + 1$ such that

$$\cos(\theta_0 + k_0\varphi_0) > 0 \qquad \text{and} \qquad \cos(\theta_0 + (k_0+1)\varphi_0) \leq 0.$$

Since the distance between two consecutive transitions from positive to negative values of cosine is at most 2π and $k_0 \geq 4m^3(3m \log m - \log \varepsilon) + 1$, we have that $k_0 - 4m^3(3m \log m - \log \varepsilon) - 1 \leq 2\pi/\varphi_0$ and by Lemma 5.5

$$k_0 \leq 4m^3(3m \log m - \log \varepsilon) + 1 + \frac{2\pi}{\varphi_0} \leq 4m^3(3m \log m - \log \varepsilon) + 1 + \frac{2\pi m^{3/2}}{\varepsilon}. \tag{20}$$

Once we have chosen k_0, the values of y_{k_0-1} and y_{k_0+2} as bounded as follows.

Lemma 5.11

$$y_{k_0-1} \geq 2|a_0|\rho_0^{k_0-1}\frac{\varphi_0}{4} \qquad \textit{and} \qquad y_{k_0+2} \leq -2|a_0|\rho_0^{k_0+2}\frac{\varphi_0}{4}.$$

Proof: We first observe that if $k_0 > 4m^3(3m \log m - \log \varepsilon) + 1$, then

$$\begin{aligned} k_0 - 1 &\geq \frac{3m \log m - \log \varepsilon}{\log(1 + 1/(4m^3))} && (\text{since } \log(1+x) \leq x) \\ &\geq \frac{(m+1)\log m + \log(4m+2) + \log(m^{3/2}/\varepsilon)}{\log(1 + 1/(4m^3))}. \end{aligned}$$

Note that since $\varepsilon \leq 1$, $\varepsilon/m^{3/2} < 1/\sqrt{3}m$ and $\varphi_0 \geq \varepsilon/m^{3/2}$ by Lemma 5.5 which implies that

$$\left(1 + \frac{1}{4m^3}\right)^{k_0-1} \geq \frac{4^{2m+1} m^{m+1}}{\varphi_0} \qquad \text{and} \qquad \frac{4^{2m} m^{m+1}}{(1 + 1/4m^3)^{k_0-1}} \leq \frac{\varphi_0}{4}.$$

In particular, if k_0 is the first index greater than $4m^3(3m \log m - \log \varepsilon)$ such that $\cos(\theta_0 + k_0\varphi_0) > 0$ and $\cos(\theta_0 + (k_0+1)\varphi_0) \leq 0$, then by Lemma 5.10

$$\begin{aligned} \frac{y_{k_0-1}}{2|a_0|\rho_0^{k_0-1}} &\geq \cos(\theta_0 + (k_0-1)\varphi_0) - \frac{4^{2m} m^{m+1}}{(1 + 1/(4m^3))^{k_0-1}} \\ &\geq \cos(\pi/2 - \varphi_0) - \frac{\varphi_0}{4} = \sin(\varphi_0) - \frac{\varphi_0}{4} \\ &\geq \frac{\varphi_0}{2} - \frac{\varphi_0}{4} = \frac{\varphi_0}{4}. \end{aligned}$$

Similarly,

$$\begin{aligned}\frac{y_{k_0+2}}{2|a_0|\rho_0^{k_0+2}} &\leq \cos(\theta_0+(k_0+2)\varphi_0)+\frac{4^{2m}m^{m+1}}{(1+1/(4m^3))^{k_0}}\\ &\leq \cos(\pi/2+\varphi_0)+\frac{\varphi_0}{4}=-\sin(\varphi_0)+\frac{\varphi_0}{4}\\ &\leq -\frac{\varphi_0}{2}+\frac{\varphi_0}{4}=-\frac{\varphi_0}{4}\end{aligned}$$

as claimed. □

With the above preparations we now can prove the main lemma.

Lemma 5.12 *If*

$$c < \frac{m^m}{(m-1)^{m-1}}-\frac{22^2m^8\log^2 m}{\log^2 D},$$

then $y_{k_0-1}>c^2$ *and* $y_{k_0+2}<0$.

Proof: $y_{k_0+2}<0$ follows directly from Lemma 5.11. Hence, we only have to show that $y_{k_0-1}>c^2$.

Step 1 We first show that a lower bound on D. If

$$c < \frac{m^m}{(m-1)^{m-1}}-\frac{22^2m^8\log^2 m}{\log^2 D},$$

then

$$\begin{aligned}\log D &> \frac{22m^4\log m}{\sqrt{m^m/(m-1)^{m-1}-c}}=\frac{22m^4\log m}{\varepsilon}\\ &\geq (12m^4\log m-4m^3\log\varepsilon)\log 3+\frac{m^4\log m}{\varepsilon} \qquad (\text{since } 1/\varepsilon>-\log\varepsilon)\\ &\geq \left(4m^3(3m\log m-\log\varepsilon)+\frac{2\pi m^{3/2}}{\varepsilon}\right)\log 3+\\ &\quad (m-1)\log(2em)+2\log(em)+\log m^{3/2}\end{aligned}$$

and, therefore,

$$D > \frac{(em)^2(2em)^{m-1}3^{k_0-1}m^{3/2}}{\varepsilon} > \frac{c^2(2em)^{m-1}3^{k_0-1}m^{3/2}}{\varepsilon} \tag{21}$$

since by Equation 20

$$k_0-1\leq 4m^3(3m\log m-\log\varepsilon)+\frac{m^{3/2}2\pi}{\varepsilon}.$$

Step 2 We now show that $y_{k_0-1} > c^2$. We have by Lemma 5.11

$$\begin{aligned} y_{k_0-1} &\geq 2|a_0|\rho_0^{k_0-1}\frac{\varphi_0}{4} \\ &\geq 2\frac{D}{(2em)^{m-1}}\rho_0^{k_0-1}\frac{\varphi_0}{4} && \text{(by Lemma 5.9)} \\ &\geq \frac{2D}{(2em)^{m-1}}(1/3)^{k_0-1}\frac{\varphi_0}{4} && \text{(since } \rho_0 \geq 1/3\text{)} \\ &\geq \frac{D\varepsilon}{(2em)^{m-1}3^{k_0-1}m^{3/2}} && \text{(by Lemma 5.5)} \\ &> c^2 && \text{(by Equation 21)} \end{aligned}$$

as claimed. □

Now that we have shown Lemma 3.5 we can reformulate Theorem 3.6 in the following way.

Theorem 5.13 *There is no search strategy for a target on m rays which is contained in the interval $[1, D]$ with a competitive ratio of less than*

$$1 + 2\left(\frac{m^m}{(m-1)^{m-1}} - \frac{22^2 m^8 \log^2 m}{\log^2 D}\right).$$

6 An Optimal Strategy

After having proven a lower bound for searching on m rays with an upper bound on the target distance, the questions remains whether this is the best bound possible. In this section we present a strategy to search on m rays that achieves a competitive ratio of $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ even if the maximum distance D of the target to the starting point is unknown. Hence, the lower bound proven in the previous section cannot be improved if we consider the converge rate as D increases to infinity. Moreover, being told an upper bound on the distance to the target is not a big advantage—even if we consider the convergence rate.

The strategy $X = (x_1, x_2, \dots)$[1] that achieves a competitive ratio of $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ is given by

$$x_i = \sqrt{1 + \frac{i}{m}}\left(\frac{m}{m-1}\right)^i.$$

The competitive ratio of Strategy X in Step $k+m$ is bounded by

$$\begin{aligned} & 1 + 2\frac{\sum_{j=1}^{k+m-1}\sqrt{1+\frac{j}{m}}\left(\frac{m}{m-1}\right)^j}{\sqrt{1+\frac{k}{m}}\left(\frac{m}{m-1}\right)^k} \\ &= 1 + 2\sum_{j=1}^{k+m-1}\sqrt{\frac{m+j}{m+k}}\left(\frac{m}{m-1}\right)^{j-k} \\ &= 1 + 2\left(\sum_{j=1}^{k-1}\sqrt{\frac{j+m}{k+m}}\left(\frac{m}{m-1}\right)^{j-k} + \sum_{j=k}^{k+m-1}\sqrt{\frac{j+m}{k+m}}\left(\frac{m}{m-1}\right)^{j-k}\right) \\ &= 1 + 2\left(\sum_{j=1}^{k-1}\sqrt{\frac{j+m}{k+m}}\left(\frac{m-1}{m}\right)^{k-j} + \sum_{j=0}^{m-1}\sqrt{1+\frac{j}{k+m}}\left(\frac{m}{m-1}\right)^{j}\right), \end{aligned}$$

[1] For convenience we start with x_1 instead of x_0.

where we assume $k \geq 2$ in the moment. We present an upper bound for the sums on the right hand side. We first consider the sum

$$\sum_{j=0}^{m-1} \sqrt{1+\frac{j}{k+m}} \left(\frac{m}{m-1}\right)^j.$$

We first observe that

$$\sqrt{1+x} \leq 1+\frac{1}{2}x,$$

for $x \leq 1$, and, therefore,

$$\begin{aligned}\sum_{j=0}^{m-1} \sqrt{1+\frac{j}{k+m}} \left(\frac{m}{m-1}\right)^j &\leq \sum_{j=0}^{m-1} \left(1+\frac{1}{2}\frac{j}{k+m}\right) \left(\frac{m}{m-1}\right)^j \\ &= \sum_{j=0}^{m-1} \left(\frac{m}{m-1}\right)^j + \frac{1}{2}\sum_{j=0}^{m-1} \frac{j}{k+m} \left(\frac{m}{m-1}\right)^j.\end{aligned}$$

The first sum is equal to

$$\sum_{j=0}^{m-1} \left(\frac{m}{m-1}\right)^j = \frac{m^m}{(m-1)^{m-1}} - (m-1) \tag{22}$$

and the second sum is equal

$$\sum_{j=0}^{m-1} \frac{j}{k+m} \left(\frac{m}{m-1}\right)^j = \frac{(m-1)m}{k+m}. \tag{23}$$

Now we consider the sum

$$\begin{aligned}\sum_{j=1}^{k-1} \sqrt{\frac{j+m}{k+m}} \left(\frac{m-1}{m}\right)^{k-j} &= \sum_{j=1}^{k-1} \sqrt{\frac{k-j+m}{k+m}} \left(\frac{m-1}{m}\right)^j \\ &= \sum_{j=1}^{k-1} \sqrt{1-\frac{j}{k+m}} \left(\frac{m-1}{m}\right)^j. \end{aligned} \tag{24}$$

Similar to above we observe that

$$\sqrt{1-x} \leq 1-\frac{1}{2}x-\frac{1}{8}x^2,$$

for $x \leq 1$, and, therefore,

$$\sum_{j=1}^{k-1} \sqrt{1-\frac{j}{k+m}} \left(\frac{m-1}{m}\right)^j \leq \sum_{j=1}^{k-1} \left(1-\frac{1}{2}\frac{j}{k+m}-\frac{1}{8}\left(\frac{j}{k+m}\right)^2\right) \left(\frac{m-1}{m}\right)^j.$$

We again compute the values of the sums on the right hand side separately.

$$\sum_{j=1}^{k-1} \left(\frac{m-1}{m}\right)^j = m-1-m\left(\frac{m-1}{m}\right)^k, \tag{25}$$

$$\sum_{j=1}^{k-1} \frac{j}{k+m} \left(\frac{m-1}{m}\right)^j = \frac{m(m-1)-(k-m-1)m\left(\frac{m-1}{m}\right)^k}{k+m}, \tag{26}$$

and

$$\sum_{j=1}^{k-1}\left(\frac{j}{k+m}\right)^2\left(\frac{m-1}{m}\right)^j = \frac{m(m-1)(2m-1)}{(k+m)^2} - \frac{(k^2+2k(m-2)+2m^2-3m+1)m\left(\frac{m-1}{m}\right)^k}{(k+m)^2}. \quad (27)$$

Hence,

$$\begin{aligned}&\sum_{j=1}^{k-1}\sqrt{\frac{j}{k+m}}\left(\frac{m-1}{m}\right)^{k-j} + \sum_{j=0}^{m-1}\sqrt{1+\frac{j}{k+m}}\left(\frac{m}{m-1}\right)^j \\ &\leq \sum_{j=1}^{k-1}\left(1-\frac{1}{2}\frac{j}{k+m}-\frac{1}{8}\left(\frac{j}{k+m}\right)\right)\left(\frac{m-1}{m}\right)^j + \sum_{j=0}^{m-1}\left(1+\frac{1}{2}\frac{j}{k+m}\right)\left(\frac{m}{m-1}\right)^j.\end{aligned}$$

Equations 22 and 25 yield

$$\begin{aligned}\sum_{j=0}^{m-1}\left(\frac{m}{m-1}\right)^j + \sum_{j=1}^{k-1}\left(\frac{m-1}{m}\right)^j &= \frac{m^m}{(m-1)^{m-1}} - (m-1) + m - 1 - m\left(\frac{m-1}{m}\right)^k \\ &= \frac{m^m}{(m-1)^{m-1}} - m\left(\frac{m-1}{m}\right)^k. \quad (28)\end{aligned}$$

Equations 23 and 26 yield

$$\begin{aligned}&\frac{1}{2}\sum_{j=0}^{m-1}\frac{j}{k+m}\left(\frac{m}{m-1}\right)^j - \frac{1}{2}\sum_{j=1}^{k-1}\frac{j}{k+m}\left(\frac{m-1}{m}\right)^j \\ &= \frac{1}{2}\frac{(m-1)m}{k+m} - \frac{1}{2}\frac{m(m-1)-(k-m-1)m\left(\frac{m-1}{m}\right)^k}{k+m} \\ &= \frac{1}{2}\frac{(k-m-1)m}{k+m}\left(\frac{m-1}{m}\right)^k. \quad (29)\end{aligned}$$

If we combine Equations 27, 28, and 29, then we obtain,

$$\begin{aligned}&\sum_{j=0}^{m-1}\sqrt{1+\frac{j}{k+m}}\left(\frac{m}{m-1}\right)^j + \sum_{j=1}^{k-1}\sqrt{\frac{j+m}{k+m}}\left(\frac{m-1}{m}\right)^{k-j} \\ &\leq \frac{m^m}{(m-1)^{m-1}} - \frac{1}{8}\frac{m(m-1)(2m-1)}{(k+m)^2} \\ &\quad + \left(\frac{1}{2}\frac{k-m-1}{k+m} + \frac{1}{8}\frac{k^2+2k(m-2)+2m^2-3m+1}{(k+m)^2} - 1\right)m\left(\frac{m-1}{m}\right)^k \\ &\leq \frac{m^m}{(m-1)^{m-1}} - \frac{1}{8}\frac{m(m-1)(2m-1)}{(k+m)^2}\end{aligned}$$

since

$$\frac{1}{2}\frac{k-m-1}{k+m} + \frac{1}{8}\frac{k^2+2k(m-2)+2m^2-3m+1}{(k+m)^2} \leq 1.$$

There are two special cases $k=1$ and $k \leq 0$ that have to be considered separately. If $k=1$, then sum 24 is 0 and sums 22 and 23 add up to

$$\frac{m^m}{(m-1)^{m-1}} - (m-1) + \frac{(m-1)m}{2(1+m)} \leq \frac{m^m}{(m-1)^{m-1}} - \frac{m-1}{2}$$

If $-m+1 \leq k \leq 0$, then the target is discovered during the first m iterations. The worst case occurs if the first $m-1$ rays are explored and then the target is detected on the mth ray at a distance of $1+\varepsilon$, for some $\varepsilon > 0$. The competitive ratio is bounded by

$$1 + 2\sum_{j=1}^{m-1} \sqrt{1+\frac{j}{m}} \left(\frac{m}{m-1}\right)^j \quad \leq \quad \frac{m^m}{(m-1)^{m-1}} - \frac{m+1}{2}.$$

Finally, we relate the number of steps $k+m$ to the distance D to the target. If the target is detected in Step $k+m$, then the distance D to s is in the interval $[\sqrt{1+\frac{k}{m}}(m/(m-1))^k, \sqrt{1+\frac{k+m}{m}}(m/(m-1))^{k+m}]$ and D is bounded from below by

$$\sqrt{1+\frac{k}{m}} \left(\frac{m}{m-1}\right)^k \leq D$$

or

$$\frac{1}{2}\log(1+k/m) + k\log\left(1+\frac{1}{m-1}\right) \leq \log D$$

which implies

$$k \leq \frac{\log D}{\log\left(1+\frac{1}{m-1}\right)} \leq (m-1)\log D.$$

Hence,

$$\begin{aligned} 1 + 2\frac{\sum_{j=1}^{k+m-1} \sqrt{1+\frac{j}{m}} \left(\frac{m}{m-1}\right)^j}{\sqrt{1+\frac{k}{m}} \left(\frac{m}{m-1}\right)^k} &\leq 1 + 2\left(\frac{m^m}{(m-1)^{m-1}} - \frac{2m-1}{8(\log D + m/(m-1))^2}\right) \\ &\leq 1 + 2\frac{m^m}{(m-1)^{m-1}} - \frac{2m-1}{4\log^2(3D)}. \end{aligned}$$

We have shown the following theorem.

Theorem 6.1 *There is a strategy X that achieves a competitive ratio of*

$$1 + 2\frac{m^m}{(m-1)^{m-1}} - \frac{2m-1}{4\log^2(3D)}$$

if the target is placed at distance $D > 1$ to s.

By Theorem 5.13 the strategy we have presented above is optimal. Note that the lower bound we have shown in Section 5 is only interesting if $\log D > 2m^4 \log m$.

7 Conclusions

We present a lower bound for the problem of searching on m concurrent rays if an upper bound D on the maximal distance to the target is given. We show that in this case the competitive ratio of a search strategy is at least $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$. Our approach is based on deriving a recursive equation for the step length in each iteration of an optimal strategy. The recursive equation gives rise to a characteristic equation whose roots determine the properties

of a strategy. By computing upper and lower bounds on the radii and polar angles of the roots in polar coordinates we can show that the competitive ratio has to be sufficiently large if the target is far away.

We also present a strategy which achieves a competitive ratio of $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ if the target is detected at distance D. The strategy does not need to know an upper bound on D in advance. Hence, the knowledge of an upper bound on the distance to the target only provides a marginal advantage to the robot—even the convergence rate is not improved.

An interesting open problem is to prove similar results for randomized strategies. One of the problems with randomized strategies is that there is no published proof that there is an optimal periodic strategy. It seems that this is a necessary step before the bounded distance problem can be attacked.

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