Visibility in Semi-convex Spaces*

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Abstract

In this paper we pursue the notion of a semi-convex space as a unifying framework for the treatment of various notions of convexity in the plane. In particular, we suggest how to capture the notion of visibility within the general framework of convexity spaces, and investigate the relationship between visibility, kernels and skulls. We prove the Kernel Theorem and the Cover Kernel Theorem, both of which relate kernels and skulls.

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1 Introduction

The concept of *visibility* plays an important role in Computational Geometry. Its origins date back to 1973 when Victor Klee posed the now famous *art gallery problem* where he asked for the minimum number of guards that are always sufficient to guard an art gallery consisting of n walls. Since then numerous visibility problems have been considered and a wealth of results has been obtained. Usually two points are defined to be visible if the line segment that is spanned by the points does not intersect the interior of an obstacle. But apart from this definition several other notions of visibility have been investigated in the past years: *staircase visibility* [CR89, MRS88], *rectangular visibility* [Kei86, MOW87, OW88] and *periscope visibility* [GN90].

In this paper we consider several definitions of visibility and propose a framework in which many of the structural aspects of "non-standard" definitions of visibility can be treated in a uniform way. In order to do so we employ an axiomatic approach based on *convexity spaces.* A convexity space is a tuple $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ where $\mathcal{C}_\mathbf{X}$ is the family of "convex sets" of the set $\mathbf{X}$. Convexity spaces represent an abstraction of the structure of convex sets in Euclidean space. Leaving aside all topological concerns we require convex sets to be closed under intersection, thus capturing their lattice-theoretic and algebraic properties. It is not surprising that such a general concept has arisen in many different contexts and has led to a number of names such as *convexity spaces, convexity structures,* or *algebraic closure systems.*

Convexity in this context is understood as a property that solely depends on the membership in $\mathcal{C}_\mathbf{X}$. Given $\mathcal{C}_\mathbf{X}$ we are led to the $\mathcal{C}_\mathbf{X}$*-hull* of a set. Two points p and q in a set $\mathbf{S}$ are said to *see* each other if the $\mathcal{C}_\mathbf{X}$-hull of p and q is contained in $\mathbf{S}$. By imposing various other conditions on the sets in $\mathcal{C}_\mathbf{X}$ we can investigate some of the structural properties of the kernel of a set; in particular, we are able to characterize the convexity spaces for which the kernel can be obtained as the intersection of the family of maximal convex sets.

The main aim of abstract convexity is to provide a unifying framework that goes beyond the concept of convexity based on line segments as in real vector spaces. In fact, there is an astonishing variety of different notions of convexity in the plane that have been considered in Computational Geometry in the past few years: *restricted orientation convexity* [Raw87], *NESW-convexity* [LP81, SW84], *rectangular convexity* [Kei86, SW84], and *geodesic convexity* [GHL*87, Tou88], to name the most prominent ones.

There is a close relationship between questions concerning visibility and those concerning convexity. In most convexity spaces the definition of a convex set is based on the property that all points of a convex set can "see" each other. So only the seeing relationship between two points has to be specified in order to define the whole convexity space.

The original motivation for an abstract treatment of convexity is to investigate the relationships between the Helly, Radon, and Carathéodory numbers in an axiomatic setting [Lev51, KW71, Sie77]. Additionally, the exchange number of a convexity space has been introduced and several generalizations of the Helly and Radon numbers have been considered [DRS81, Sie82].

The observation that many convexity spaces can be decomposed into simpler components has led to a second approach. Its main goal is to deduce properties of the convexity space from those of its components. All the results obtained by this approach are concerned with the characteristics of the convex hull operator [Deg79, Eck68, RWar, Sie81]. Little attention has been paid to problems which involve visibility.

On the other hand, visibility is a well studied concept in the context of real vector spaces [Bre82, Tor88, Val65]. But, as the above mentioned examples illustrate, this is often too restrictive a setting. So, in order to define and investigate notions of visibility that are not based on the properties of straight lines, it is desirable to take the more general approach of convexity spaces and make use of general results obtained in this context. This leads to a number of questions which have not been considered in the framework of real vector spaces.

As it turns out there is an especially suited variant of convexity spaces to treat questions that arise in the context of visibility. We term this variant *semi-convex spaces*.

The rest of this paper is organized as follows. The next section contains some of the definitions and conventions we will use throughout the rest of this paper. In Section 3 we introduce the concept of convexity spaces and give the motivation for the definition of semi-convex spaces. We then look at the notion of visibility in this abstract framework. This is followed in Section 4, the main section of this paper, with the presentation of our results on kernels and skulls in semi-convex spaces. It culminates in the proofs of the Kernel Theorem and the Cover Kernel Theorem. Section 5 is concerned with different methods to define semi-convex spaces. In Section 6 we consider a number of examples of convexity and semi-convex spaces. Finally, in Section we prove that semi-convex spaces that are induced by certain metrics in the plane satisfy the Kernel Theorem.

2 Definitions and Notation

In this section we give short definitions of the basic concepts we will make use of throughout the rest of this paper.

2.1 Topology

We start off with some notations of elementary topology. We are only concerned with the topology of the plane $\mathbb{E}^2$ which is based on the family of norms $\|\cdot\|_p$ where $p \in \mathbb{R}_+$. It is defined as follows. If $x = (x_1, x_2)$ is a point in the plane, then the $\|\cdot\|_p$-norm of x is

$$\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}.$$

$\|\cdot\|_p$ induces the *L_p-metric (or d_p-metric)* which is defined by $L_p(x,y) = \|x-y\|_p$ where x and y are two points in the plane. Note that all norms in $\mathbb{E}^2$ induce the same topology. We will only consider the L_2-metric (d_2-metric)—which we also call the *Euclidean metric*—and the L_1-metric.

The interior of a set $\mathbf{S}$ is denoted by $int(\mathbf{S})$ and the closure of $\mathbf{S}$ by $\overline{\mathbf{S}}$. We denote the *boundary* of $\mathbf{S}$ by $\partial\mathbf{S}$.

Another important concept are *curves* or *paths*. A *curve* or *path* is an equivalence class $\mathcal{C}$ of continuous mappings $\varphi : I \longrightarrow I\!\!E^2$ where I is a real interval. $\varphi \in \mathcal{C}$ is called a *parametrization of* $\mathcal{C}$. We will mostly consider a curve as a point set but it should be kept in mind that a curve can always be endowed with a parametrization. Given two curves $\mathcal{P}$ and $\mathcal{Q}$ with two parametrizations $\varphi_{\mathcal{P}} : \langle a, b] \longrightarrow I\!\!E^2$ and $\varphi_{\mathcal{Q}} : [c, d\rangle \longrightarrow I\!\!E^2$ (the brackets $\langle$ and $\rangle$ are used to indicate that it doesn't matter if the left or right end point belongs to the interval or not) such that $\varphi_{\mathcal{P}}(b) = \varphi_{\mathcal{Q}}(c)$ we define the *concatenation of* $\mathcal{P}$ *and* $\mathcal{Q}$, denoted by $\mathcal{P} * \mathcal{Q}$, as the equivalence class of the mapping $\varphi_{\mathcal{P}*\mathcal{Q}} : \langle a, d - c + b\rangle \longrightarrow I\!\!E^2$ which is given by

$$\varphi_{\mathcal{P}*\mathcal{Q}}(x) = \begin{cases} \varphi_{\mathcal{P}}(x) & \text{if } x \in \langle a, b] \\ \varphi_{\mathcal{Q}}(x + c - b) & \text{if } x \in [b, d - c + b\rangle. \end{cases}$$

We say that a set $\mathbf{S}$ is *path-connected* if, for any two points x and y in $\mathbf{S}$, there exists a path $\mathcal{P}$ connecting x and y that is completely contained in $\mathbf{S}$. The sets we will consider in this paper are usually compact, path-connected sets. More precisely, they are the domain of a simple curve as defined below.

2.2 Jordan Curves

A curve $\mathcal{C}$ is called *simple* if there exists a parametrization $\varphi : \langle a, b\rangle \longrightarrow I\!\!E^2$ of $\mathcal{C}$ that is injective on $[a, b)$. We say $\mathcal{C}$ is closed if there exists a parametrization φ of $\mathcal{C}$ over $[0, 1]$ such that $\varphi(0) = \varphi(1)$. We call a simple closed curve according to [Rin75] a *Jordan curve*. By the famous Jordan Curve Theorem [Rin75, 40.1] a simple closed curve $\mathcal{J}$ partitions the plane into two parts, one bounded called *the interior of* $\mathcal{J}$ and denoted by $int(\mathcal{J})$, and the other unbounded called *the exterior of* $\mathcal{J}$ and denoted by $ext(\mathcal{J})$. We define the *domain* of $\mathcal{J}$ to be the union of $int(\mathcal{J})$ and $\mathcal{J}$ and denote it by $dom(\mathcal{J})$.

2.3 Polygons

Polygons are a special case of simple curves. Usually a *polygon* $\mathcal{P}$ is defined to be a closed curve that consists of a finite number of line segments, i.e., there is a parametrization of $\mathcal{P}$ that is piecewise linear. The *edges* of a polygon $\mathcal{P}$ are the maximal line segments it consists of. The *vertices* of a simple polygon are defined to be the intersection points of the edges.

Since we are mostly interested in the properties of the domain of a polygon instead of the curve itself, we will also use the term polygon to denote the *domain* of a polygon. In order to distinguish between curves and domains, we use capital calligraphic letters like $\mathcal{P}$ for the first and capital bold-face letters like $\mathbf{P}$ for the latter.[1] So whenever we want to refer to the curve that encloses a given domain $\mathbf{D}$ we will speak of the *boundary of* $\mathbf{D}$. If

[1] Note that calligraphic letters may also be used to denote a family of sets.

we speak of the vertices and edges of $\mathbf{P}$, we, of course, mean the vertices and edges of the boundary of $\mathbf{P}$.

3 Semi-convex Spaces and Visibility

This section deals with the fundamental definitions upon which our investigations of convexity and semi-convex spaces are based. We first define what we understand by a semi-convex space and relate it to the better known convexity space structure. Then, we establish some of the properties of the convex hull operator in semi-convex spaces; a suitable notion of visibility is abstracted and some of its consequences are illustrated. Finally, we deal with skulls and aligned spaces.

3.1 Convexity and Semi-convex Spaces

We base the following investigation on the concept of a semi-convex space. But before we introduce it, we want to look at the more common structure of a convexity space whose formal definition was first given by Levi [Lev51]. A convexity space is intended to abstract some of the essential properties of convex sets in n-dimensional Euclidean space.

Definition 3.1 *Let* $\mathbf{X}$ *be a set and* $\mathcal{C}_{\mathbf{X}}$ *be a collection of subsets of* $\mathbf{X}$*. Then,* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is a* convexity space *if*

1. $\emptyset$ *and* $\mathbf{X}$ *are in* $\mathcal{C}_{\mathbf{X}}$*; and*
2. *for all* $\mathcal{C} \subseteq \mathcal{C}_{\mathbf{X}}$*, we have* $\bigcap \mathcal{C} \in \mathcal{C}_{\mathbf{X}}$*, where by* $\bigcap \mathcal{C}$ *we mean* $\bigcap_{\mathbf{C} \in \mathcal{C}} \mathbf{C}$.[2]

$\mathbf{X}$ is called the *groundset* of the convexity space and $\mathcal{C}_{\mathbf{X}}$ contains the "convex sets" of $\mathbf{X}$. In order to stress the analogy to Euclidean vector spaces we will call the elements of $\mathbf{X}$ *points*. Each set in $\mathcal{C}_{\mathbf{X}}$ is called $\mathcal{C}_{\mathbf{X}}$-*convex* (or convex for short if the convexity space is understood).[3] So, the only characteristic required of convex sets is their closure under intersection. It is obvious that additional properties are needed to generalize the more intricate properties of $\mathbb{E}^n$ since a wide variety of structures satisfies the above definition. Simple examples are:

1. *The discrete convexity space* $(\mathbf{X}, 2^{\mathbf{X}})$, where $2^{\mathbf{X}}$ is the powerset of $\mathbf{X}$. Obviously, any intersection of subsets of $\mathbf{X}$ is again a subset of $\mathbf{X}$.
2. *The topological convexity space.* If $(\mathbf{X}, \mathcal{T})$ is a topological space, then $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ with $\mathcal{C}_{\mathbf{X}} = \{\mathbf{C} \subseteq \mathbf{X} \mid \mathbf{C} \text{ is closed in } (\mathbf{X}, \mathcal{T})\}$ is a convexity space since the intersection of closed sets remains closed.

[2]In this paper we will use $\bigcap \mathcal{F}$ to denote the intersection of all sets in a family $\mathcal{F}$ and $\bigcup \mathcal{F}$ to denote the union of all sets in $\mathcal{F}$.

[3]Note that the convexity of a set is solely determined by membership in $\mathcal{C}_{\mathbf{X}}$ and is not implied by any other property.

Immediately associated with a convexity space is the convex hull operator.

Definition 3.2 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a convexity space and* $\mathbf{Y}$ *a subset of* $\mathbf{X}$*. We define the* $\mathcal{C}_{\mathbf{X}}$-hull *of* $\mathbf{Y}$ *as* $\bigcap\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\}$ *and denote it by* $\mathcal{C}_{\mathbf{X}}$*-hull*$(\mathbf{Y})$*, i.e.,*

$$\mathcal{C}_{\mathbf{X}}\textit{-hull}(\mathbf{Y}) = \bigcap\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\}.$$

It is easy to see that the $\mathcal{C}_{\mathbf{X}}$*-hull* operator is well defined and unique.

Given two distinct points p and q in the Euclidean plane, their convex hull is the line segment joining them. Since this is the basis of of Euclidean visibility, our abstract definition of visibility is analogous.

Definition 3.3 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a convexity space and* $\mathbf{Y} \subseteq \mathbf{X}$*. We say that two points* x *and* y *in* $\mathbf{Y}$ see each other *if* $\mathcal{C}_{\mathbf{X}}$*-hull*$(\{x, y\}) \subseteq \mathbf{Y}$*. We write* x $\mathcal{C}_{\mathbf{Y}}$*-sees* y *in this case.*[4]

Observe that $\mathcal{C}_{\mathbf{Y}}$*-sees* is symmetric, but not necessarily reflexive or transitive. We have reflexivity, for all $\mathbf{Y} \subseteq \mathbf{X}$, if and only if all singletons in $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ are convex. We give a special name to convexity or semi-convex spaces that have this property.

Definition 3.4 *A convexity space (semi-convex space)* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is called* simple *if, for all* $x \in \mathbf{X}$*, we have* $\{x\} \in \mathcal{C}_{\mathbf{X}}$*.*

We now contrast convexity spaces with the (very similar) definition of a semi-convex space.

Definition 3.5 *Let* $\mathbf{X}$ *be a set and* $\mathcal{C}_{\mathbf{X}}$ *be a collection of subsets of* $\mathbf{X}$*. Then,* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is a* semi-convex space *if*

1. $\emptyset$ *is in* $\mathcal{C}_{\mathbf{X}}$*; and*
2. *for all* $\mathcal{C} \subseteq \mathcal{C}_{\mathbf{X}}$*, we have* $\bigcap \mathcal{C} \in \mathcal{C}_{\mathbf{X}}$*.*

At first glance there seems to be no fundamental difference between a semi-convex space and a convexity space. A semi-convex space $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ can be easily transformed into a convexity space just by adding $\mathbf{X}$ to $\mathcal{C}_{\mathbf{X}}$. So the question arises why this class of spaces should be considered instead of the better understood class of convexity spaces. But taking a closer look reveals important characteristics of semi-convex spaces that are not present in convexity spaces.

It is our aim to investigate the notion of visibility that arises in the abstract framework outlined above. From the viewpoint of visibility, a convexity space is trivial since every point can see all the others. Hence, if we want to investigate visibility relations, we have to consider subsets of the convexity space endowed with the following structure. Let $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ be a convexity space, $\mathbf{Y}$ a subset of $\mathbf{X}$, and $\mathcal{C}_{\mathbf{Y}} = \{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{C} \subseteq \mathbf{Y}\}$. The visibility structure of $\mathbf{Y}$ is solely determined by the family $\mathcal{C}_{\mathbf{Y}}$ since two points see each other if and

[4]This notation will be motivated later.

only if there is a $\mathbf{C} \in \mathcal{C}_{\mathbf{Y}}$ that contains them. Unfortunately, $(\mathbf{Y}, \mathcal{C}_{\mathbf{Y}})$ is not a convexity space unless $\mathbf{Y} \in \mathcal{C}_{\mathbf{X}}$ and this is, of course, the trivial case. But it is easy to see that $(\mathbf{Y}, \mathcal{C}_{\mathbf{Y}})$ is a semi-convex space. Thus, semi-convex spaces seem the right setting for our investigations concerning abstract visibility. Since any convexity space is also a semi-convex space, the results we obtain also apply to convexity spaces.

Observe that we then have that semi-convex spaces are closed under subspace formation if we consider the subspaces defined above. Also note that this does not necessarily mean that our approach is more general than just considering convexity spaces. Some of the questions we consider here will not arise in the theory of convexity spaces or can be trivially answered.

Furthermore, note that there are many ways to endow a subset $\mathbf{Y}$ of a convexity space $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ with a convexity structure. If we were interested in the hull properties, we might look at the family $\mathcal{C}'_{\mathbf{Y}} = \{\mathbf{C} \cap \mathbf{Y} \mid \mathbf{C} \in \mathcal{C}_{\mathbf{X}}\}$ since we have, for all $\mathbf{Z} \subseteq \mathbf{Y}$, that $\mathcal{C}'_{\mathbf{Y}}\text{-}hull(\mathbf{Z}) = \mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{P}) \cap \mathbf{Y}$. $(\mathbf{Y}, \mathcal{C}'_{\mathbf{Y}})$ is again a convexity space but we loose all visibility information.

Before we start with some more definitions that concern star-shapedness and kernels of sets, we need to explore some of the properties of the hull-operator in semi-convex spaces which behaves quite differently from the hull-operator in convexity spaces.

3.2 The Hull-Operator in Semi-convex Spaces

The definition of the convex hull of a set $\mathbf{Y} \subseteq \mathbf{X}$ in the semi-convex space $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is exactly the same as the hull-definition for convexity spaces, i.e.,

$$\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y}) = \bigcap \{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\}.$$

But since we do not necessarily have that $\mathbf{X}$ is in $\mathcal{C}_{\mathbf{X}}$, it may be that $\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\}$ is empty. In topology one usually defines $\bigcap \emptyset = \mathbf{X}$ since it is assumed that any point x satisfies $x \in \bigcap \emptyset$. We will deviate from this tradition. If $\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\} = \emptyset$, then this means that $\mathbf{Y}$ contains points that cannot see each other in $\mathbf{X}$. The $\mathcal{C}_{\mathbf{X}}$-*hull* of $\mathbf{Y}$ does not exist and we define $\bigcap \emptyset = \emptyset$. This also conforms with our requirement that, for all $\mathcal{C} \subseteq \mathcal{C}_{\mathbf{X}}$, we have $\bigcap \mathcal{C} \in \mathcal{C}_{\mathbf{X}}$.

The $\mathcal{C}_{\mathbf{X}}$-*hull* operator has the following properties.

Theorem 3.1 *If $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is a semi-convex space, then, for all $\mathbf{Z}, \mathbf{Y} \subseteq \mathbf{X}$ with $\mathbf{Z} \subseteq \mathbf{Y}$, we have*

(i) $\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y}) \in \mathcal{C}_{\mathbf{X}}$;

(ii) $\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y}) = \mathbf{Y}$ if and only if $\mathbf{Y} \in \mathcal{C}_{\mathbf{X}}$;

(iii) if $\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y}) \neq \emptyset$, then $\mathbf{Y} \subseteq \mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})$;

(iv) if $\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y}) \neq \emptyset$, then $\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Z}) \subseteq \mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})$;

(v) if $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Z}) = \emptyset$ and $\mathbf{Z} \neq \emptyset$, then $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y}) = \emptyset$;[5]

(vi) $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y})) = \mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y})$.

Proof: We will prove the claims in the sequence in which they are stated.

(i) This property is just the closure property of $\mathcal{C}_{\mathbf{X}}$ under intersection.

(ii) This property is a consequence of Property (i) and the fact that $\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\}$ contains $\mathbf{Y}$ if $\mathbf{Y} \in \mathcal{C}_{\mathbf{X}}$.

(iii) If $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y}) \neq \emptyset$, then $\mathcal{C} = \{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\}$ is non-empty and since all the sets in $\mathcal{C}$ contain $\mathbf{Y}$, $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y}) = \bigcap \mathcal{C}$ also contains $\mathbf{Y}$.

(iv) If $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y}) \neq \emptyset$, then $\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\}$ is non-empty and since $\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\} \subseteq \{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Z} \subseteq \mathbf{C}\}$, the claim follows immediately from the definition of the $\mathcal{C}_{\mathbf{X}}$-*hull*.

(v) If $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Z}) = \emptyset$, then we also have $\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Z} \subseteq \mathbf{C}\} = \emptyset$ since $\mathbf{Z} \neq \emptyset$. By the above inclusion $\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\} \subseteq \{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Z} \subseteq \mathbf{C}\}$ the claim again follows immediately.

(vi) If $\mathbf{C} = \mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y})$ is non-empty, then by Property (ii) we have $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y})) = \mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{C}) = \mathbf{C}$ since $\mathbf{C} \in \mathcal{C}_{\mathbf{X}}$. On the other hand, if $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y}) = \emptyset$, then $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y})) = \mathcal{C}_{\mathbf{X}}$-*hull*$(\emptyset) = \emptyset = \mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y})$.

$\square$

Corollary 3.2 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space and* $\emptyset \neq \mathbf{Y} \subseteq \mathbf{X}$. $\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\} = \emptyset$ *if and only if* $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y}) = \emptyset$.

As in the case of convexity spaces there is an inverse of the Theorem 3.1, i.e., a hull-operator that satisfies all the properties listed gives rise to semi-convex space.

Theorem 3.3 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space and* φ *be a mapping* $\varphi : 2^{\mathbf{X}} \longrightarrow 2^{\mathbf{X}}$ *where* $2^{\mathbf{X}}$ *denotes the power set of* $\mathbf{X}$. *If* φ *satisfies the following conditions, for all* $\mathbf{Y}, \mathbf{Z} \subseteq \mathbf{X}$ *with* $\mathbf{Z} \subseteq \mathbf{Y}$*,*

(i) $\varphi(\emptyset) = \emptyset$;

(ii) if $\varphi(\mathbf{Y}) \neq \emptyset$, then $\mathbf{Y} \subseteq \varphi(\mathbf{Y})$;

(iii) if $\varphi(\mathbf{Y}) \neq \emptyset$, then $\varphi(\mathbf{Z}) \subseteq \varphi(\mathbf{Y})$;

(iv) if $\varphi(\mathbf{Z}) = \emptyset$ and $\mathbf{Z} \neq \emptyset$, then $\varphi(\mathbf{Y}) = \emptyset$; and

[5]Note that Property (iv) and Property (v) can be stated as one condition for $\mathbf{Z} \neq \emptyset$: if $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y}) \neq \emptyset$, then $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Z}) \neq \emptyset$ and $\mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Z}) \subseteq \mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{Y})$.

(v) $\varphi(\varphi(\mathbf{Y})) = \varphi(\mathbf{Y})$,[6]

then $\mathcal{C}_{\mathbf{X}} = \{\mathbf{C} \subseteq \mathbf{X} \mid \varphi(\mathbf{C}) = \mathbf{C}\}$ *is closed under intersection,* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is a semispace and the* $\mathcal{C}_{\mathbf{X}}$*-hull-operator of* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *equals* φ*.*

Proof: First we prove the closure of $\mathcal{C}_{\mathbf{X}}$ under intersection. To this end let $\mathcal{C} \subseteq \mathcal{C}_{\mathbf{X}}$. We have to show that $\varphi(\bigcap \mathcal{C}) = \bigcap \mathcal{C}$. If $\bigcap \mathcal{C} = \emptyset$, then we clearly have $\bigcap \mathcal{C} \in \mathcal{C}_{\mathbf{X}}$ by Property (i). So suppose $\bigcap \mathcal{C} \neq \emptyset$.

Since $\bigcap \mathcal{C}$ is non-empty, we have that $\varphi(\mathbf{C}) \neq \emptyset$, for all $\mathbf{C} \in \mathcal{C}$ by Property (iii), and, hence, $\varphi(\bigcap \mathcal{C}) \subseteq \bigcap_{\mathbf{C} \in \mathcal{C}} \varphi(\mathbf{C}) = \bigcap \mathcal{C}$. To see the reverse inclusion we first assume that $\varphi(\bigcap \mathcal{C}) = \emptyset$. Then, we have that $\varphi(\mathbf{C}) = \emptyset$, for all $\mathbf{C} \supseteq \bigcap \mathcal{C}$ by Property (iv). But since $\bigcap \mathcal{C} \subseteq \mathbf{C}$, for all $\mathbf{C} \in \mathcal{C}$, we obtain $\mathbf{C} = \varphi(\mathbf{C}) = \emptyset$ which contradicts $\bigcap \mathcal{C} \neq \emptyset$. Hence, we have $\varphi(\bigcap \mathcal{C}) \neq \emptyset$ and by Property (ii) we have $\bigcap \mathcal{C} \subseteq \varphi(\bigcap \mathcal{C})$. Therefore, $\bigcap \mathcal{C} = \varphi(\bigcap \mathcal{C})$ and $\bigcap \mathcal{C} \in \mathcal{C}_{\mathbf{X}}$. This proves the first part of the claim.

Since $\varphi(\emptyset) = \emptyset$, we have that $\emptyset \in \mathcal{C}_{\mathbf{X}}$ and $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is a semi-convex space. Now consider the $\mathcal{C}_{\mathbf{X}}$-*hull* of $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ and let $\mathbf{Y}$ be a subset of $\mathbf{X}$. We want to show that $\varphi(\mathbf{Y}) = \mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})$. Since the claim obviously holds for the empty set, we assume that $\mathbf{Y}$ is non-empty. First suppose that $\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\} \neq \emptyset$. Since $\mathbf{C}' = \mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})$ is in $\mathcal{C}_{\mathbf{X}}$, we have $\varphi(\mathbf{C}') = \mathbf{C}'$ and $\mathbf{C}' \supseteq \mathbf{Y}$. Property (iii) then implies that $\varphi(\mathbf{C}') = \mathbf{C}' \supseteq \varphi(\mathbf{Y})$ and Property (iv) yields that $\varphi(\mathbf{Y}) \neq \emptyset$. This proves $\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y}) \supseteq \varphi(\mathbf{Y})$.

To see the reverse inclusion we have by Property (ii) that $\varphi(\mathbf{Y}) \in \{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\}$ and, thus, $\varphi(\mathbf{Y}) \supseteq \mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})$. Therefore, $\varphi(\mathbf{Y}) = \mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})$.

Now assume $\{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{Y} \subseteq \mathbf{C}\} = \emptyset$. Then, $\varphi(\mathbf{Y}) \not\supseteq \mathbf{Y}$ and by Property (ii) we have $\varphi(\mathbf{Y}) = \emptyset = \mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})$. □

Since, for two points x, y in $\mathbf{X}$, $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x, y\}) = \emptyset$ indicates that two points cannot see each other *even in* $\mathbf{X}$, we need to change our definition of $\mathcal{C}_{\mathbf{Y}}$-visibility in semi-convex spaces.

Definition 3.6 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space and* $\mathbf{Y} \subseteq \mathbf{X}$*. We say that two points* x *and* y *in* $\mathbf{Y}$ $\mathcal{C}_{\mathbf{Y}}$-see each other *if* $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x, y\}) \neq \emptyset$ *and* $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x, y\}) \subseteq \mathbf{Y}$*.*

A more concise statement can be made using the definition of $\mathcal{C}_{\mathbf{Y}}$ and the $\mathcal{C}_{\mathbf{Y}}$-*hull*. x $\mathcal{C}_{\mathbf{Y}}$-*sees* y in $\mathbf{Y}$ iff $\mathcal{C}_{\mathbf{Y}}\text{-}hull(\{x, y\}) \neq \emptyset$. Note that this definition reduces to Definition 3.3 in convexity spaces.

3.3 Visibility in Semi-convex Spaces

Once having established a consistent definition of visibility it is easy to generalize the notions of star shaped sets and kernels for semi-convex spaces which we need in the remainder of this paper.

Definition 3.7 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space and* $\mathbf{Y} \subseteq \mathbf{X}$*.*

[6]Note that this does not imply Property (i) since there maybe no set $\mathbf{Y} \neq \emptyset$ with $\varphi(\mathbf{Y}) = \emptyset$.

1. *For $x \in \mathbf{Y}$, we define $\mathcal{C}_{\mathbf{X}}$-star$(x, \mathbf{Y}) = \{y \in \mathbf{Y} \mid x \; \mathcal{C}_{\mathbf{Y}}$-sees $y\}$.*

2. *$\mathbf{Y}$ is* star-shaped *if $\mathbf{Y} = \mathcal{C}_{\mathbf{X}}$-star$(y, \mathbf{Y})$, for some $y \in \mathbf{Y}$.*

3. *$\mathcal{C}_{\mathbf{X}}$-kernel$(\mathbf{Y}) = \{y \in \mathbf{Y} \mid \mathcal{C}_{\mathbf{X}}$-star$(y, \mathbf{Y}) = \mathbf{Y}\}$.*

Note that, in general, $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y})$ is not convex. As an example we introduce the following semi-convex space.

Example Let $\mathbf{X} = I\!N$ and $\mathcal{C}_{\mathbf{X}} = \{\mathbf{C} \subseteq I\!N \mid \mathbf{C} \text{ is finite}\}$. Now consider $\mathbf{Y}$ = set of odd numbers and let $x, y \in \mathbf{Y}$. Since $\{x, y\}$ is finite and, hence, convex, we get $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, y\}) = \{x, y\}$ and this is surely contained in $\mathbf{Y}$. Therefore, all $x, y \in \mathbf{Y}$ can see each other by our above definition of visibility and $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y}) = \mathbf{Y}$. But $\mathbf{Y}$ is not finite and, therefore, $\mathbf{Y}$ is not convex, i.e., $\mathbf{Y} \notin \mathcal{C}_{\mathbf{X}}$.

Because $\mathcal{C}_{\mathbf{X}}$-*hull* is an expansive operator if it is non-empty (Statement (ii) of Theorem 3.1), we deduce immediately that, for a non-empty convex set $\mathbf{C}$ and any two points x and y in $\mathbf{C}$, $\emptyset \neq \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, y\}) \subseteq \mathcal{C}_{\mathbf{X}}$-*hull*$(\mathbf{C}) = \mathbf{C}$. Thus, for all points x and y in $\mathbf{C}$, we have x $\mathcal{C}_{\mathbf{C}}$-*sees* y. The converse does not hold in general as the above example illustrates. This kind of unexpected behaviour of semi-convex spaces leads to the concept of *visibility convexity.*

Definition 3.8 *Let $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ be a semi-convex space and $\mathbf{C} \subseteq \mathbf{X}$. We say $\mathbf{C}$ is ($\mathcal{C}_{\mathbf{X}}$-)*visibility convex (all-seeing convex) *if, for all $x, y \in \mathbf{C}$, we have $\mathcal{C}_{\mathbf{X}}$-hull$(\{x, y\}) \neq \emptyset$ and $\mathcal{C}_{\mathbf{X}}$-hull$(\{x, y\}) \subseteq \mathbf{C}$; in other words, $\mathcal{C}_{\mathbf{X}}$-kernel$(\mathbf{C}) = \mathbf{C}$.*

Definition 3.9 *A semi-convex space $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is said to be* complete *if all visibility convex sets are convex.*

Hence, in a complete semi-convex space, all-seeingness and convexity are equivalent notions. This property makes complete semi-convex spaces a natural setting for questions concerning visibility. Moreover, most of the semi-convex spaces that arise in practice are based on the definition of the convex hull of two points and extend this definition to a complete semi-convex space. As an example consider the definition of convexity in the plane. A set $\mathbf{C}$ is convex if it contains the line segment between any two points x and y in $\mathbf{C}$; that is, if all x and y in $\mathbf{C}$ can see each other.

Although not all semi-convex spaces are complete, we can always embed an incomplete semi-convex space in a complete one, in a natural way, as we now demonstrate.

Definition 3.10 *Let $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ be a semi-convex space. The* completion *of $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is denoted by $(\mathbf{X}, \tilde{\mathcal{C}}_{\mathbf{X}})$ and is defined by*

$$\tilde{\mathcal{C}}_{\mathbf{X}} = \{\mathbf{C} \subseteq \mathbf{X} \mid \mathbf{C} \textit{ is } \mathcal{C}_{\mathbf{X}}\textit{-visibility convex}\}.$$

Clearly, we have $\mathcal{C}_{\mathbf{X}} \subseteq \tilde{\mathcal{C}}_{\mathbf{X}}$.[7]

[7] Note that the emptyset also satisfies Definition 3.10. Hence, $\emptyset \in \tilde{\mathcal{C}}_{\mathbf{X}}$.

Theorem 3.4 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space. Then, the completion* $(\mathbf{X}, \tilde{\mathcal{C}}_{\mathbf{X}})$ *of* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is a semi-convex space and it is complete.*

Proof: 1. We begin by proving that $\tilde{\mathcal{C}}_{\mathbf{X}}$ is closed under intersection, since this already suffices to guarantee that $(\mathbf{X}, \tilde{\mathcal{C}}_{\mathbf{X}})$ is a semi-convex space by the above observation. So take a non-empty family of convex set $\tilde{\mathcal{C}} \subseteq \tilde{\mathcal{C}}_{\mathbf{X}}$. We want to show that $\bigcap \tilde{\mathcal{C}}$ is $\mathcal{C}_{\mathbf{X}}$-visibility convex. If $\bigcap \tilde{\mathcal{C}} = \emptyset$, we are done. So assume $\bigcap \tilde{\mathcal{C}} \neq \emptyset$ and choose any two points $x, y \in \bigcap \tilde{\mathcal{C}}$. Since, for all $\tilde{\mathbf{C}} \in \tilde{\mathcal{C}}$, we have $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \tilde{\mathbf{C}}$, it follows immediately that $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \bigcap \tilde{\mathcal{C}}$.

It only remains to show that also $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \neq \emptyset$. Since there is a least one non-empty $\tilde{\mathbf{C}} \in \tilde{\mathcal{C}}$ with $\{x,y\} \subseteq \tilde{\mathbf{C}}$, we have $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \neq \emptyset$ by the definition of $\tilde{\mathcal{C}}_{\mathbf{X}}$ and $\bigcap \tilde{\mathcal{C}}$ is $\mathcal{C}_{\mathbf{X}}$-visibility convex. Hence, $\bigcap \tilde{\mathcal{C}} \in \tilde{\mathcal{C}}_{\mathbf{X}}$.

2. We next show that $(\mathbf{X}, \tilde{\mathcal{C}}_{\mathbf{X}})$ is complete, that is, if, for all $x, y \in \tilde{\mathbf{C}}$, $\tilde{\mathcal{C}}_{\mathbf{X}}\text{-}hull(\{x,y\}) \neq \emptyset$ and $\tilde{\mathcal{C}}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \tilde{\mathbf{C}}$, then $\tilde{\mathbf{C}} \in \tilde{\mathcal{C}}_{\mathbf{X}}$. To this end we prove that, for all $x, y \in \mathbf{X}$, $\tilde{\mathcal{C}}_{\mathbf{X}}\text{-}hull(\{x,y\}) = \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\})$ and the claim follows from the definition of $\tilde{\mathcal{C}}_{\mathbf{X}}$.

First consider the case $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \neq \emptyset$. Then, we have $\emptyset \neq \{\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \mid \{x,y\} \subseteq \mathbf{C}\} \subseteq \{\tilde{\mathbf{C}} \in \tilde{\mathcal{C}}_{\mathbf{X}} \mid \{x,y\} \subseteq \tilde{\mathbf{C}}\}$ and, hence, $\tilde{\mathcal{C}}_{\mathbf{X}}\text{-}hull(\{x,y\}) \neq \emptyset$ by Corollary 3.2. Since $\tilde{\mathcal{C}}_{\mathbf{X}} \supseteq \mathcal{C}_{\mathbf{X}}$ it follows immediately that $\tilde{\mathcal{C}}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\})$. To see the reverse inclusion observe that a set $\tilde{\mathbf{C}}$ is $\tilde{\mathcal{C}}_{\mathbf{X}}$-convex only if, for all $u, v \in \tilde{\mathbf{C}}$, $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{u,v\}) \neq \emptyset$ and $\tilde{\mathbf{C}}$ contains $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{u,v\})$. But, this implies that $\tilde{\mathcal{C}}_{\mathbf{X}}\text{-}hull(\{x,y\})$ contains $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\})$.

If $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) = \emptyset$, then there is no visibility convex set $\tilde{\mathbf{C}}$ that contains x and y and, therefore, $\tilde{\mathcal{C}}_{\mathbf{X}}\text{-}hull(\{x,y\}) = \emptyset$. □

Corollary 3.5 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space. The completion* $(\mathbf{X}, \tilde{\mathcal{C}}_{\mathbf{X}})$ *of* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is the smallest complete space that contains* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$.

Proof: Let $(\mathbf{X}, \bar{\mathcal{C}}_{\mathbf{X}})$ be a complete semi-convex space that contains $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ and $\mathbf{C}$ a set with $\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{C}) = \mathbf{C}$. Since $\bar{\mathcal{C}}_{\mathbf{X}}$ contains $\mathcal{C}_{\mathbf{X}}$, we have $\bar{\mathcal{C}}_{\mathbf{X}}\text{-}kernel(\mathbf{C}) = \mathbf{C}$ and $\mathbf{C} \in \bar{\mathcal{C}}_{\mathbf{X}}$. Hence, $\tilde{\mathcal{C}}_{\mathbf{X}} \subseteq \bar{\mathcal{C}}_{\mathbf{X}}$. □

As another visibility-related concept we need the notion of a *join.*

Definition 3.11 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space,* $\mathbf{C} \in \mathcal{C}_{\mathbf{X}}$*, and* $x \in \mathbf{X}$*, we define the* $\mathcal{C}_{\mathbf{X}}$-join *of* x *and* $\mathbf{C}$ *by*

$$\mathcal{C}_{\mathbf{X}}\text{-}join(x, \mathbf{C}) = \begin{cases} \bigcup_{c \in \mathbf{C}} \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,c\}) & \text{if } \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,c\}) \neq \emptyset,\ \forall c \in \mathbf{C} \\ \emptyset & \text{otherwise.} \end{cases}$$

The join of a convex set $\mathbf{C}$ and a point x consists, intuitively speaking, of all the line segments between x and points c in $\mathbf{C}$. It is easy to show that the join in the plane is always convex if we consider normal convexity. This, however, is not true for arbitrary semi-convex spaces—we will look at some examples in the next section—which leads to our next definition.

Definition 3.12 *Let* $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ *be a semi-convex space.* $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ *is said to satisfy the* join condition *if, for all* $x \in \mathbf{X}$ *and* $\mathbf{C} \in \mathcal{C}_\mathbf{X} \setminus \{\emptyset\}$*, we have*

$$\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C}) \textit{ is convex.}$$

Note that if $\mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C}) \neq \emptyset$, then $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C})$ is non-empty and the inclusion $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C}) \subseteq \mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C})$ holds for any semi-convex space while the reverse inclusion only holds for semi-convex spaces that satisfy the join condition. This is stated in the following lemma.

Lemma 3.6 *Let* $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ *be a semi-convex space.* $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ *satisfies the join condition if and only if, for all* $x \in \mathbf{X}$ *and* $\mathbf{C} \in \mathcal{C}_\mathbf{X} \setminus \{\emptyset\}$*, we have that* $\mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C}) = \mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C})$.

Proof: If $\mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C}) = \mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C})$, for all $x \in \mathbf{X}$ and $\mathbf{C} \in \mathcal{C}_\mathbf{X} \setminus \{\emptyset\}$, then $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ clearly satisfies the join condition. So suppose that $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ satisfies the join property. We want to show that $\mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C}) = \mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C})$. Hence, let x be a point in $\mathbf{X}$ and $\mathbf{C}$ a convex set in $\mathcal{C}_\mathbf{X} \setminus \{\emptyset\}$. Since $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ satisfies the join property, we have that $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C}) \in \mathcal{C}_\mathbf{X}$.

First suppose that $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C})$ is non-empty. Since $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C})$ is a convex set that contains x and $\mathbf{C}$, we have that the family of sets $\{\mathbf{C}' \in \mathcal{C}_\mathbf{X} \mid \{x\} \cup \mathbf{C} \subseteq \mathbf{C}'\}$ is non-empty and, therefore, $\mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C}) \neq \emptyset$ by Corollary 3.2. This implies that $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C})$ contains $\mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C})$. On the other hand, on the other hand we have that, for all $c \in \mathbf{C}$, $\mathcal{C}_\mathbf{X}\text{-}hull(\{x, c\}) \neq \emptyset$ and $\mathcal{C}_\mathbf{X}\text{-}hull(\{x, c\}) \subseteq \mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C})$. Since $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C}) = \bigcup_{c \in \mathbf{C}} \mathcal{C}_\mathbf{X}\text{-}hull(\{x, c\})$, we obtain $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C}) \subseteq \mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C})$. Hence, $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C}) = \mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C})$.

If $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C}) = \emptyset$, then there is at least one $c \in \mathbf{C}$ with $\mathcal{C}_\mathbf{X}\text{-}hull(\{x, c\}) = \emptyset$. The inclusion property of the $\mathcal{C}_\mathbf{X}\text{-}hull$ then implies that $\mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C}) = \emptyset$. Hence, we have in both cases that $\mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C}) = \mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C})$ which proves the claim. □

Now the reason why the empty set is excluded from the definition of the join condition becomes apparent. For we have that $\mathcal{C}_\mathbf{X}\text{-}join(x, \emptyset) = \bigcup_{c \in \emptyset} \mathcal{C}_\mathbf{X}\text{-}hull(\{x, c\}) = \emptyset \neq \mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \emptyset)$ if there is a least one point $x \in \mathbf{X}$ with a non-empty hull.

While we have that in convexity spaces the inclusion $\mathcal{C}_\mathbf{X}\text{-}join(x, \mathbf{C}) \subseteq \mathcal{C}_\mathbf{X}\text{-}hull(\{x\} \cup \mathbf{C})$ always holds, this is not the case in semi-convex spaces as the next example illustrates.

Example Let the groundset $\mathbf{X}$ be defined as the set $\{1, 2, 3, 4, 5\}$ and the family of convex sets $\mathcal{C}_\mathbf{X}$ be given by $\emptyset$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 5\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1\}$. Then, we have that $\mathcal{C}_\mathbf{X}\text{-}join(2, \{1, 3, 4\}) = \{1, 2\} \cup \{1, 2, 3\} \cup \{1, 2, 4\} = \{1, 2, 3, 4\}$ and this is not a subset of $\mathcal{C}_\mathbf{X}\text{-}hull(\{1, 2, 3, 4\}) = \emptyset$.

If we consider complete semi-convex spaces (which is what we will do in most cases) the join condition can be reduced to the following three-point condition.

Theorem 3.7 *If* $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ *is a complete semi-convex space, then the following two conditions are equivalent.*

1. *For all points x, y, and z in $\mathbf{X}$, $\mathcal{C}_{\mathbf{X}}$-join$(x, \mathcal{C}_{\mathbf{X}}$-hull$(\{y, z\}))$ is convex.*

2. *For all points x in $\mathbf{X}$ and for all sets $\mathbf{C}$ in $\mathcal{C}_{\mathbf{X}}$, $\mathcal{C}_{\mathbf{X}}$-join$(x, \mathbf{C})$ is convex.*

Proof: $1 \Rightarrow 2$. Consider some point x in $\mathbf{X}$ and some $\mathbf{C}$ in $\mathcal{C}_{\mathbf{X}}$. We denote $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C})$ by $\mathbf{Q}$. We assume that $\mathbf{Q}$ is non-empty since if $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C}) = \emptyset$, then $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C})$ is convex and we are done. For every pair y_1 and y_2 of points in $\mathbf{Q}$, we have to show that $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{y_1, y_2\}) \neq \emptyset$ and $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{y_1, y_2\}) \subseteq \mathbf{Q}$. Because $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is complete, this implies that $\mathbf{Q}$ is convex.

So let $y_1, y_2 \in \mathbf{Q}$. Since $\mathbf{Q} \neq \emptyset$, there are two points c_1 and c_2 in $\mathbf{C}$ with $y_i \in \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, c_i\})$, for $i = 1, 2$. Because c_1 and c_2 are in $\mathbf{C}$, we have that $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{c_1, c_2\})$ is non-empty and $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{c_1, c_2\}) \subseteq \mathbf{C} \subseteq \mathbf{Q}$. Since $\mathbf{Q} \neq \emptyset$, we further have that $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, y\}) \neq \emptyset$, for any $y \in \mathcal{C}_{\mathbf{X}}$-*hull*$(\{c_1, c_2\})$ and, therefore, $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathcal{C}_{\mathbf{X}}$-*hull*$(\{c_1, c_2\}))$ is non-empty and contains y_1 and y_2. But, $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathcal{C}_{\mathbf{X}}$-*hull*$(\{c_1, c_2\}))$ is convex, by assumption; hence, we have that $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{y_1, y_2\}) \subseteq \mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathcal{C}_{\mathbf{X}}$-*hull*$(\{c_1, c_2\})) \subseteq \mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C}) = \mathbf{Q}$ and $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{y_1, y_2\}) \neq \emptyset$.

$2 \Rightarrow 1$. Trivial. □

3.4 Skulls and Aligned Semi-convex spaces

It is the main aim of this paper to relate the maximal convex subsets of a set to its kernel. We also denote maximal convex subsets as "skulls".

Definition 3.13 *Let $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ be a semi-convex space and $\mathbf{Y} \subseteq \mathbf{X}$. Then,*

1. $\mathbf{S} \subseteq \mathbf{Y}$ *is a* $\mathcal{C}_{\mathbf{X}}$-skull *of* $\mathbf{Y}$ *if* $\mathbf{S} \in \mathcal{C}_{\mathbf{X}}$ *and there is no* $\mathbf{S}' \in \mathcal{C}_{\mathbf{X}}$ *such that* $\mathbf{S} \subset \mathbf{S}' \subseteq \mathbf{Y}$.

2. *We denote the family $\{\mathbf{S} \subseteq \mathbf{Y} \mid \mathbf{S}$ is a $\mathcal{C}_{\mathbf{X}}$-skull of $\mathbf{Y}\}$ of all skulls in $\mathbf{Y}$ by $\mathcal{C}_{\mathbf{X}}$-skulls$(\mathbf{Y})$.*

As we have seen before, semi-convex spaces can behave in a rather unexpected manner. This also applies to the definition of skulls. Given a set $\mathbf{Y} \subseteq \mathbf{X}$ in a semi-convex space $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ and a convex subset $\mathbf{C}$ of $\mathbf{Y}$, we cannot always assume that there is a $\mathcal{C}_{\mathbf{X}}$-*skull* $\mathbf{S}$ of $\mathbf{Y}$ that contains $\mathbf{C}$ as the next example illustrates.

Example Let $\mathbf{X}$ be the closed set of reals $[0, 1]$ and let $\mathcal{C}_{\mathbf{X}}$ be the family of all closed intervals in $\mathbf{X}$. Let $\mathbf{Y}$ be the open interval $(0, 1)$. As it can be seen easily, $\mathbf{Y}$ contains convex sets, $[1/4, 3/4]$ for instance, but there is no maximal convex set in $\mathbf{Y}$. For, consider any convex set $[\alpha, \beta] \subseteq \mathbf{Y}$; it is contained in $\mathbf{Y}$ and, therefore, $\alpha > 0$. Now $[\alpha, \beta] \subset [\alpha/2, \beta] \subset \mathbf{Y}$ and $[\alpha, \beta]$ cannot be maximal.

We say a semi-convex space is a *space with skulls* if, for all subsets $\mathbf{Y}$ of $\mathbf{X}$ and all convex sets $\mathbf{C} \subseteq \mathbf{Y}$, there is a skull $\mathbf{S} \in \mathcal{C}_{\mathbf{X}}$-*skulls*$(\mathbf{Y})$ that contains $\mathbf{C}$. The above example suggests that we may also require that the union of nested chains[8] of convex sets to be convex as well.

Definition 3.14 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space. We call* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *an* aligned (semi-convex) space *if, for every nested chain* $\mathcal{N} \subseteq \mathcal{C}_{\mathbf{X}}$*, the union of* $\mathcal{N}$ *is also convex; that is,* $\bigcup \mathcal{N} \in \mathcal{C}_{\mathbf{X}}$.

Aligned spaces are well studied objects in the literature [Ham63, Jam82, Sie84] but usually for quite different reasons than those stated here. The following lemma shows that the existence of skulls is, indeed, guaranteed by forcing the union of nested families to be convex.

Lemma 3.8 *If* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is an aligned space, then we have that, for all* $\mathbf{Y} \subseteq \mathbf{X}$ *and for all* $\mathbf{C} \in \mathcal{C}_{\mathbf{X}}$ *with* $\mathbf{C} \subseteq \mathbf{Y}$*, there is a* $\mathcal{C}_{\mathbf{X}}$*-skull* $\mathbf{S}$ *of* $\mathbf{Y}$ *that contains* $\mathbf{C}$.

Proof: To see this, take any $\mathbf{Y} \subseteq \mathbf{X}$ and $\mathbf{C}_0 \in \mathcal{C}_{\mathbf{X}} \setminus \{\emptyset\}$ with $\mathbf{C}_0 \subseteq \mathbf{Y}$.

Let $\mathcal{S}$ be the family of all convex sets in $\mathbf{Y}$ that contain $\mathbf{C}_0$. Note that $\mathcal{S}$ is non-empty. $\mathcal{S}$ is a partially-ordered set and any nested chain $\mathcal{N} \subseteq \mathcal{S}$ has an upper bound $\bigcup \mathcal{N} \in \mathcal{C}_{\mathbf{X}}$ with $\mathbf{C}_0 \subseteq \bigcup \mathcal{N} \subseteq \mathbf{Y}$, since $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is an aligned space. By Zorn's lemma, there is a maximal element $\mathbf{S}$ in $\mathcal{S}$; clearly then, $\mathbf{S}$ is a $\mathcal{C}_{\mathbf{X}}$-*skull* of $\mathbf{Y}$ containing $\mathbf{C}$. □

Another counter-intuitive observation is that $\mathcal{C}_{\mathbf{X}}$-*skulls*$(\mathbf{Y})$ need not necessarily cover $\mathbf{Y}$, in the sense that $\bigcup \mathcal{C}_{\mathbf{X}}$-*skulls*$(\mathbf{Y})$ is not necessarily $\mathbf{Y}$ itself, but only a subset. Whenever an aligned space is simple, however, we obtain a cover since every point is a convex subset and, therefore, there exists a maximal convex subset containing it.

We have introduced the notions of complete and aligned spaces above. Fortunately, there is a straightforward relationship between them.

Lemma 3.9 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space. If* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is complete, then* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is an aligned space.*

Proof: Let $\mathcal{N}$ be a nested chain of convex sets and let $\mathbf{C} = \bigcup \mathcal{N}$. W.l.o.g. assume that $\mathbf{C} \neq \emptyset$. Further, let x, y be arbitrary points in $\mathbf{C}$. It suffices to show that x and y see each other in $\mathbf{C}$ to prove that $\mathbf{C}$ is convex because $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is complete. For x and y there are two convex sets $\mathbf{C}_x$ and $\mathbf{C}_y$ in $\mathcal{N}$ such that $x \in \mathbf{C}_x$ and $y \in \mathbf{C}_y$. Since $\mathcal{N}$ is a nested chain we can assume that $\mathbf{C}_x \subseteq \mathbf{C}_y$. So $\mathbf{C}_x \cup \mathbf{C}_y = \mathbf{C}_y$ is convex, and $\emptyset \neq \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, y\}) \subseteq \mathbf{C}_y \subseteq \mathbf{C}$. □

So whenever we are dealing with complete semi-convex spaces Lemma 3.9 ensures that skulls automatically exist. There is also a (weak) converse that requires the join condition (see Theorem 2 in Kay and Womble [KW71] for a different proof where aligned convexity spaces are considered).

[8] A nested chain $\mathcal{N}$ is a family of sets that is totally ordered w.r.t. inclusion.

Lemma 3.10 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space with skulls. If* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *satisfies the join condition, then* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is complete.*

Proof: Consider a set $\mathbf{C} \subseteq \mathbf{X}$ which is visibility convex. We prove that $\mathbf{C}$ is convex, i.e., $\mathbf{C} \in \mathcal{C}_{\mathbf{X}}$. Since $\mathbf{C}$ is non-empty, it contains at least one point x, and because $\emptyset \neq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x, x\}) \subseteq \mathbf{C}$, by assumption, there is a $\mathcal{C}_{\mathbf{X}}$-*skull* $\mathbf{S}_x$ in $\mathbf{C}$ with $x \in \mathbf{S}_x$. Now, take a point $y \in \mathbf{C}$; then, we have $\emptyset \neq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{s, y\}) \subseteq \mathbf{C}$, for all points $s \in \mathbf{S}_x$, since y can see all the points in $\mathbf{C}$. In other words, $\mathcal{C}_{\mathbf{X}}\text{-}join(y, \mathbf{S}_x) \subseteq \mathbf{C}$.

But, $\mathcal{C}_{\mathbf{X}}\text{-}join(y, \mathbf{S}_x)$ is non-empty since it is contained in $\mathbf{C}$ by assumption and convex. By the definition of a $\mathcal{C}_{\mathbf{X}}$-*skull* this implies that $\mathcal{C}_{\mathbf{X}}\text{-}join(y, \mathbf{S}_x) = \mathbf{S}_x$. Thus, $y \in \mathbf{S}_x$ and $\mathbf{S}_x = \mathbf{C}$; therefore, $\mathbf{C}$ is convex. □

4 The Kernel Theorem

This section contains the main results of this paper. We state and prove the Kernel Theorem which gives a complete characterization of those semi-convex spaces for which the kernel of a set $\mathbf{Y}$ is given by the intersection of all skulls in $\mathbf{Y}$; hereby, we make crucial use of the concepts introduced so far. The proof of the theorem is partially due to G. Rawlins [Raw87, Theorem 6.3.1] who gives a proof of the "if" part in the case of convexity spaces under slightly stronger assumptions. Unfortunately, he fails to recognize that the existence of skulls has to be required and, hence, has no analog to Condition 1.

Theorem 4.1 (Kernel Theorem) *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space. Then, we have, for all* $\mathbf{Y} \subseteq \mathbf{X}$,

$$\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y}) = \bigcap \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$$

if and only if either $\mathcal{C}_{\mathbf{X}} = \{\emptyset\}$ *or the following four conditions hold:*

(i) $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is a semi-convex space with skulls;

(ii) for all $x \in \mathbf{X}$ and for all $\mathbf{C} \in \mathcal{C}_{\mathbf{X}}$, $\mathcal{C}_{\mathbf{X}}\text{-}join(x, \mathbf{C})$ is convex;

(iii) for all $x \in \mathbf{X}$, $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x\}) \neq \emptyset$;

(iv) for all x and y in $\mathbf{X}$, we have $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y\}) \subseteq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x\}) \cup \{y\}$.

Proof: The claim clearly holds for $\mathcal{C}_{\mathbf{X}} = \{\emptyset\}$, so we immediately turn to the case $\mathcal{C}_{\mathbf{X}} \neq \{\emptyset\}$. Let $K = \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y})$ and $I = \bigcap \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$.

if. We split the proof into two parts.

$K \subseteq I$. If $K = \emptyset$, this holds vacuously, so assume that $K \neq \emptyset$. Consider $x \in K$; we prove that $x \in I$. Let $\mathbf{S}$ be a skull in $\mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$ and s a point in $\mathbf{S}$; since $x \in \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y})$, we have x $\mathcal{C}_{\mathbf{Y}}$-*sees* s. Thus, $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x, s\})$ is non-empty and contained in $\mathbf{Y}$. Since s is arbitrary, we have that $\mathbf{S} \subseteq \mathcal{C}_{\mathbf{X}}\text{-}join(x, \mathbf{S}) =$

$\bigcup_{s\in\mathbf{S}} \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,s\}) \subseteq \mathbf{Y}$; furthermore, $\mathcal{C}_{\mathbf{X}}\text{-}join(x,\mathbf{S})$ is convex, by assumption. But $\mathbf{S}$ is a maximal inscribed convex set of $\mathbf{Y}$; therefore, $\mathcal{C}_{\mathbf{X}}\text{-}join(x,\mathbf{S}) = \mathbf{S}$, $x \in \mathbf{S}$, and hence $x \in I$.

$I \subseteq K$**.** Again assume that $I \neq \emptyset$ and consider $x \in I$ and an arbitrary point $y \in \mathbf{Y}$. We have to show that x $\mathcal{C}_{\mathbf{Y}}$*-sees* y. Since $\emptyset \neq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x\}) \subseteq I \subseteq \mathbf{Y}$ and I is convex and non-empty, we have $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y\}) \subseteq \{y\} \cup \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x\}) \subseteq \mathbf{Y}$ by Condition (iv) and $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y\}) \neq \emptyset$ by Condition (iii). Furthermore, since $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y\}) \subseteq \mathbf{Y}$, we know that there is an $\mathbf{S}_y \in \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$ with $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y\}) \subseteq \mathbf{S}_y$. Now $x \in I \subseteq \mathbf{S}_y$ and, thus, $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \mathbf{S}_y \subseteq \mathbf{Y}$ and $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \neq \emptyset$. Therefore, x $\mathcal{C}_{\mathbf{Y}}$*-sees* y and $x \in K$.

only if. Assume that $\mathcal{C}_{\mathbf{X}} \neq \{\emptyset\}$ and that, for all $\mathbf{Y} \subseteq \mathbf{X}$, we have that $\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y})$ equals $\bigcap \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$. We prove that $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ satisfies the four conditions of the statement.

Condition i: We have to show that skulls exist in $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$. In order to do so we make use of Lemma 3.8 and show that $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is aligned. For this let $\mathcal{N} \subseteq \mathcal{C}_{\mathbf{X}}$ be a nested chain of convex sets and let $\mathbf{Q}$ denote $\bigcup \mathcal{N}$. Consider $x, y \in \mathbf{Q}$. As in the proof of Lemma 3.9, we obtain a set $\mathbf{C} \in \mathcal{N}$ with $x, y \in \mathbf{C}$; thus $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \mathbf{C} \subseteq \mathbf{Q}$ and $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \neq \emptyset$. Hence, $\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Q}) = \mathbf{Q}$ and since $\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Q})$ is convex (it is the intersection of convex sets), we have $\mathbf{Q} \in \mathcal{C}_{\mathbf{X}}$.

Condition ii: We now prove that the $\mathcal{C}_{\mathbf{X}}\text{-}join$ is convex. Consider an arbitrary point $x \in \mathbf{X}$ and an arbitrary $\mathbf{C} \in \mathcal{C}_{\mathbf{X}} \setminus \{\emptyset\}$; we show that $\mathcal{C}_{\mathbf{X}}\text{-}join(x,\mathbf{C})$ which we denote by $\mathbf{Y}$ is convex. W.l.o.g. assume that $\mathbf{Y} \neq \emptyset$. For all $y \in \mathbf{Y}$, we have $y \in \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,c\}) \neq \emptyset$, for some $c \in \mathbf{C}$. But, this implies that $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \neq \emptyset$ and $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,c\}) \subseteq \mathbf{Y}$; hence, x $\mathcal{C}_{\mathbf{Y}}$*-sees* y. In other words, $x \in \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y})$ and, thus, $x \in \bigcap \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$. Therefore, all skulls of $\mathbf{Y}$ contain x. Because $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is an aligned space as we proved above, there is a skull $\mathbf{S}$ of $\mathbf{Y}$ that contains $\mathbf{C}$. $\mathbf{S}$ also contains x so $\mathbf{Y} = \mathcal{C}_{\mathbf{X}}\text{-}join(x,\mathbf{C}) \subseteq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x\} \cup \mathbf{C}) \subseteq \mathbf{S} \subseteq \mathbf{Y}$ and $\mathbf{Y}$ is convex.

Condition iii: Assume there is an $x \in \mathbf{X}$ with $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x\}) = \emptyset$ or, in other words, $x \notin \mathbf{C}$, for all $\mathbf{C} \in \mathcal{C}_{\mathbf{X}}$. Since $\mathcal{C}_{\mathbf{X}} \neq \{\emptyset\}$, there is a $\mathbf{C} \in \mathcal{C}_{\mathbf{X}}$ with $\mathbf{C} \neq \emptyset$. Let $\mathbf{Y} = \{x\} \cup \mathbf{C}$. Since x is not contained in a convex set, $\mathbf{Y}$ is not in $\mathcal{C}_{\mathbf{X}}$ and, in particular, $\mathbf{Y} \neq \mathbf{C}$. Therefore, $\mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y}) = \{\mathbf{C}\}$ and $\bigcap \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y}) = \mathbf{C}$. But no point in $\mathbf{Y}$ can see x and, hence, $\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y}) = \emptyset$ which contradicts our assumption.

Condition iv: Consider two arbitrary points x and y in $\mathbf{X}$ and let $\mathbf{Y} = \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x\}) \cup \{y\}$. We prove that $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y\}) \subseteq \mathbf{Y}$ by contradiction.

Assume that $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y\}) \not\subseteq \mathbf{Y}$. Then, $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y,z\}) \not\subseteq \mathbf{Y}$ or $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y,z\}) = \emptyset$, for all $z \in \mathbf{Y}$, and, hence, y is not visible from any point in $\mathbf{Y}$ (including itself). Thus, $\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y}) = \emptyset$.

We have by similar reasoning that, for every $\mathbf{S} \in \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$, y is not in $\mathbf{S}$. But since $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x\}) \neq \emptyset$, $\mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$ contain at least one skull $\mathbf{S}$ with

$\mathbf{S} \supseteq \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x\})$. Both statements together imply that $\mathbf{S} = \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x\})$ and $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x\})$ is the only skull of $\mathbf{Y}$. Thus, $\bigcap \mathcal{C}_{\mathbf{X}}$-*skulls*$(\mathbf{Y}) = \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x\}) \neq \emptyset$ and we have a contradiction; so $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{y\}) \subseteq \mathbf{Y}$.

□

As an immediate consequence we get the following corollary.

Corollary 4.2 *If* $(\mathbf{Y}, \mathcal{C}_{\mathbf{Y}})$ *is a semi-convex space that satisfies the conditions of the Kernel Theorem, then* $\mathcal{C}_{\mathbf{X}}$*-kernel*$(\mathbf{Y})$ *is convex, for all* $\mathbf{Y} \subseteq \mathbf{X}$.

We are now in a position to give a more elaborate example which will illustrate the application of the above concepts in the case of *partition convexity.*

Example Let $\mathbf{X}$ be some non-empty set and $\Pi = \{\mathbf{X}_i \subseteq \mathbf{X} \mid i \in I\}$ be a partition of $\mathbf{X}$ ($\mathbf{X} = \bigcup \Pi$ and $\mathbf{X}_i \cap \mathbf{X}_j \neq \emptyset$ implies $i = j$). Then, $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is a semi-convex space, where $\mathcal{C}_{\mathbf{X}} = \{\emptyset\} \cup \Pi$, because, for a family $\mathcal{C} \subseteq \mathcal{C}_{\mathbf{X}}$, we have that

$$\bigcap \mathcal{C} = \begin{cases} \emptyset & \text{if } \emptyset \in \mathcal{C} \text{ or } \mathbf{X}_i \text{ and } \mathbf{X}_j \in \mathcal{C} \text{ with } i \neq j. \\ \mathbf{X}_i & \text{if } \mathcal{C} \text{ is } \{\mathbf{X}_i\}. \end{cases}$$

Consider a non-empty subset $\mathbf{Y}$ of $\mathbf{X}$ such that $\mathbf{Y} \notin \mathcal{C}_{\mathbf{X}}$. We show that $\mathbf{Y}$ is not visibility convex. If $\mathbf{Y} \subset \mathbf{X}_i$, for some $\mathbf{X}_i \in \Pi$ and $x \in \mathbf{Y}$, then $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x\}) = \mathbf{X}_i \neq \mathbf{Y}$. If there are two points x and y in $\mathbf{Y}$ with $x \in \mathbf{X}_i$ and $y \in \mathbf{X}_j$, $i \neq j$, then $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, y\}) = \emptyset$. In any case there are points x and y in $\mathbf{Y}$ that do not see each other. Hence, $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is complete.

If there is an $\mathbf{X}_i$ that consists of at least two points, say x and y, then consider $\mathbf{Q} = \mathbf{X}_j \cup \{x\}$, for $\mathbf{X}_j \neq \mathbf{X}_i$. Immediately, we obtain that $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Q}) = \emptyset$ and $\mathcal{C}_{\mathbf{X}}$-*skulls*$(\mathbf{Q}) = \{\mathbf{X}_j\}$; hence, $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ does not satisfy the conditions of the Kernel Theorem.

Otherwise, each $\mathbf{X}_i$ is a singleton set and $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is simple which, in particular, implies that $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ satisfies Conditions (iii) and (iv) of the Kernel Theorem. Furthermore, any nested chain of convex sets consists of at most three elements, the empty set, a singleton, and $\mathbf{X}$; hence, $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is aligned. Now consider three points x, y, and $z \in \mathbf{X}$. If $x = y = z$, then $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathcal{C}_{\mathbf{X}}$-*hull*$(\{y, z\})) = \{x\}$, otherwise $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathcal{C}_{\mathbf{X}}$-*hull*$(\{y, z\})) = \emptyset$. In both cases, the join is convex; thus, the Kernel Theorem holds in $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$.

4.1 Near-simple Semi-convex Spaces

Conditions (iii) and (iv) of the Kernel Theorem seem to be (and are) very artificial. This is due to the fact that we wanted to *characterize* the semi-convex spaces that satisfy the kernel-skull relation. From now on we will call a semi-convex space that satisfies Conditions (iii) and (iv) of the Kernel Theorem *near-simple.* It is the main aim of this section to justify this terminology, i.e., to show that there is a very close relationship between near-simple and simple semi-convex spaces. We start off with a slight reformulation of near-simplicity.

Lemma 4.3 *Let* $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ *be a semi-convex space. We have, for all points* x *and* $y \in \mathbf{X}$, $\mathcal{C}_\mathbf{X}$*-hull*$(\{x\}) \neq \emptyset$ *and* $\mathcal{C}_\mathbf{X}$*-hull*$(\{x\}) \subseteq \mathcal{C}_\mathbf{X}$*-hull*$(\{y\}) \cup \{x\}$ *if and only if there exists a set* $\mathbf{C}_0 \in \mathcal{C}_\mathbf{X}$ *such that, for all* $x \in \mathbf{X}$, $\mathcal{C}_\mathbf{X}$*-hull*$(\{x\}) = \{x\} \cup \mathbf{C}_0$.

Proof: Let $x \in \mathbf{X}$. Obviously, if a set $\mathbf{C}_0$ exists such that $\mathcal{C}_\mathbf{X}$-*hull*$(\{x\}) = \{x\} \cup \mathbf{C}_0$, for all $x \in \mathbf{X}$, then we have, for all $x, y \in \mathbf{X}$, that $\mathcal{C}_\mathbf{X}$-*hull*$(\{x\}) \neq \emptyset$ and $\mathcal{C}_\mathbf{X}$-*hull*$(\{y\}) = \{y\} \cup \mathbf{C}_0 \subseteq \{x\} \cup \mathbf{C}_0 \cup \{y\} = \mathcal{C}_\mathbf{X}$-*hull*$(\{x\}) \cup \{y\}$.

Therefore, let us assume that we have, for all $x, y \in \mathbf{X}$, that $\mathcal{C}_\mathbf{X}$-*hull*$(\{x\})$ is non-empty and that $\mathcal{C}_\mathbf{X}$-*hull*$(\{y\}) \subseteq \mathcal{C}_\mathbf{X}$-*hull*$(\{x\}) \cup \{y\}$; we define $\mathbf{C}_0 = \bigcap_{x \in \mathbf{X}} \mathcal{C}_\mathbf{X}$-*hull*$(\{x\})$. Let x_0 be a fixed point in $\mathbf{X}$ and consider $\mathcal{C}_\mathbf{X}$-*hull*$(\{x_0\})$. Since $\mathcal{C}_\mathbf{X}$-*hull*$(\{x_0\}) \subseteq \mathcal{C}_\mathbf{X}$-*hull*$(\{y\}) \cup \{x_0\}$, for *all* $y \in \mathbf{X}$, we have $\mathcal{C}_\mathbf{X}$-*hull*$(\{x_0\}) \subseteq \bigcap_{y \in \mathbf{X}} \mathcal{C}_\mathbf{X}$-*hull*$(\{y\}) \cup \{x_0\} = \mathbf{C}_0 \cup \{x_0\}$. On the other hand, $\mathbf{C}_0$ is a subset of $\mathcal{C}_\mathbf{X}$-*hull*$(\{x_0\})$ as the intersection of all singleton hulls. Finally, we have $x_0 \in \mathcal{C}_\mathbf{X}$-*hull*$(\{x_0\})$ because $\mathcal{C}_\mathbf{X}$-*hull*$(\{x_0\}) \neq \emptyset$, and the claimed equality follows. $\square$

The above lemma asserts that a near-simple semi-convex space $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ is simple up to a constant additional point set. There is a canonical simple semi-convex space $(\mathbf{X}', \mathcal{C}'_{\mathbf{X}'})$ associated with $(\mathbf{X}, \mathcal{C}_\mathbf{X})$.

Definition 4.1 *Let* $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ *be a near-simple semi-convex space,* $\mathbf{X}' = \mathbf{X} \setminus \mathbf{C}_0$, *and* $\mathcal{C}'_{\mathbf{X}'} = \{\mathbf{C}' \subseteq \mathbf{X} \mid \mathbf{C}' = \mathbf{C} \setminus \mathbf{C}_0, \mathbf{C} \in \mathcal{C}_\mathbf{X}\}$ *where* $\mathbf{C}_0 = \bigcap_{x \in \mathbf{X}} \mathcal{C}_\mathbf{X}$*-hull*$(\{x\})$. *We call* $(\mathbf{X}', \mathcal{C}'_{\mathbf{X}'})$ *the* simplification *of* $(\mathbf{X}, \mathcal{C}_\mathbf{X})$.

The next lemma shows that the above definition is, indeed, reasonable.

Lemma 4.4 *If* $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ *is a near-simple semi-convex space, then* $(\mathbf{X}', \mathcal{C}'_{\mathbf{X}'})$ *is a simple semi-convex space.*

Proof: First, we have to check that $(\mathbf{X}', \mathcal{C}'_{\mathbf{X}'})$ is a semi-convex space. Obviously, we have that $\mathbf{C}' \subseteq \mathbf{X}'$, for all $\mathbf{C}' \subseteq \mathcal{C}'_{\mathbf{X}'}$. Now let $\mathcal{C}'$ be a subfamily of $\mathcal{C}'_{\mathbf{X}'}$ and $\bar{\mathcal{C}}$ the family $\{\mathbf{C} \in \mathcal{C}_\mathbf{X} \mid \exists \mathbf{C}' \in \mathcal{C}' \text{ with } \mathbf{C}' = \mathbf{C} \setminus \mathbf{C}_0\}$. We have that $\bigcap \mathcal{C}' = \bigcap_{\mathbf{C} \in \bar{\mathcal{C}}} (\mathbf{C} \setminus \mathbf{C}_0)$. With $\bar{\mathbf{C}} = \bigcap \bar{\mathcal{C}} \in \mathcal{C}_\mathbf{X}$, we have $\bigcap \mathcal{C}' = \bar{\mathbf{C}} \setminus \mathbf{C}_0 \in \mathcal{C}'_{\mathbf{X}'}$.

Further, we claim that $(\mathbf{X}', \mathcal{C}'_{\mathbf{X}'})$ is simple. Let $x \in \mathbf{X}'$ and consider x an element of $\mathbf{X}$. Since $\mathcal{C}_\mathbf{X}$-*hull*$(\{x\}) = \{x\} \cup \mathbf{C}_0 \in \mathcal{C}_\mathbf{X}$, we have $\{x\} \in \mathcal{C}'_{\mathbf{X}'}$ and $(\mathbf{X}', \mathcal{C}'_{\mathbf{X}'})$ is simple. $\square$

The most important property of the simplification of a near-simple semi-convex space is that all the conditions we have investigated in this paper carry over. But before we can prove this we need a technical lemma.

Lemma 4.5 *Let* $(\mathbf{X}, \mathcal{C}_\mathbf{X})$ *be a near-simple semi-convex space with* $\mathbf{C}_0 = \bigcap_{x \in \mathbf{X}} \mathcal{C}_\mathbf{X}$*-hull*$(\{x\})$ *and* $(\mathbf{X}', \mathcal{C}'_{\mathbf{X}'})$ *its simplification. If* $\mathbf{Y} \subseteq \mathbf{X}$, *then*

$$\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\mathbf{Y} \setminus \mathbf{C}_0) = \mathcal{C}_\mathbf{X}\text{-}hull(\mathbf{Y}) \setminus \mathbf{C}_0. \qquad (*)$$

In particular, if $\mathcal{C}_\mathbf{X}$*-hull*$(\mathbf{Y}) \neq \emptyset$, *then* $\mathcal{C}'_{\mathbf{X}'}$*-hull*$(\mathbf{Y} \setminus \mathbf{C}_0) \cup \mathbf{C}_0 = \mathcal{C}_\mathbf{X}$*-hull*$(\mathbf{Y})$.

Proof: First assume that $\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\mathbf{Y}\setminus\mathbf{C}_0)\neq\emptyset$. By definition we have $\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\mathbf{Y}\setminus\mathbf{C}_0)=\bigcap\{\mathbf{C}'\in\mathcal{C}'_{\mathbf{X}'}\mid\mathbf{Y}\setminus\mathbf{C}_0\subseteq\mathbf{C}'\}$. Since $\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\mathbf{Y}\setminus\mathbf{C}_0)\cap\mathbf{C}_0=\emptyset$, we have

$$\begin{aligned}
\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\mathbf{Y}\setminus\mathbf{C}_0)\cup\mathbf{C}_0 &= \bigcap\{\mathbf{C}'\in\mathcal{C}'_{\mathbf{X}'}\mid\mathbf{Y}\setminus\mathbf{C}_0\subseteq\mathbf{C}'\}\cup\mathbf{C}_0\\
&= \bigcap\{\mathbf{C}'\cup\mathbf{C}_0\mid\mathbf{Y}\setminus\mathbf{C}_0\subseteq\mathbf{C}'\text{ and }\mathbf{C}'\in\mathcal{C}'_{\mathbf{X}'}\}\\
&= \bigcap\{\mathbf{C}\in\mathcal{C}_{\mathbf{X}}\mid\mathbf{Y}\subseteq\mathbf{C}\} \qquad (**)\\
&= \mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})
\end{aligned}$$

Equality $(**)$ follows from the fact that every non-empty set $\mathbf{C}$ in $\mathcal{C}_{\mathbf{X}}$ is of the form $\mathbf{C}=\mathbf{C}'\cup\mathbf{C}_0$ with $\mathbf{C}'\in\mathcal{C}'_{\mathbf{X}'}$.

If $\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\mathbf{Y}\setminus\mathbf{C}_0)=\emptyset$, then $\{\mathbf{C}\in\mathcal{C}'_{\mathbf{X}'}\mid\mathbf{Y}\setminus\mathbf{C}_0\subseteq\mathbf{C}\}=\emptyset$ by Corollary 3.2 and, hence, $\{\mathbf{C}\in\mathcal{C}_{\mathbf{X}}\mid\mathbf{Y}\subseteq\mathbf{C}\}=\emptyset$ which implies that $\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})=\emptyset$.

The second claim follows from the fact, that if $\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})\neq\emptyset$, then there is a $x\in\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})$ and $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x\})=\{x\}\cup\mathbf{C}_0\subseteq\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})$. Since by the first claim $\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})\supseteq\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\mathbf{Y}\setminus\mathbf{C}_0)$, we have $\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\mathbf{Y}\setminus\mathbf{C}_0)\cup\mathbf{C}_0\subseteq\mathcal{C}_{\mathbf{X}}\text{-}hull(\mathbf{Y})$. The reverse inclusion again follows by $(*)$. □

Now we can state the main result about the simplification of a semi-convex space.

Theorem 4.6 *Let $(\mathbf{X},\mathcal{C}_{\mathbf{X}})$ be a near-simple semi-convex space and $(\mathbf{X}',\mathcal{C}'_{\mathbf{X}'})$ its simplification. If $(\mathbf{X},\mathcal{C}_{\mathbf{X}})$*

(i) is aligned or

(ii) is complete or

(iii) satisfies the join-condition,

then $(\mathbf{X}',\mathcal{C}'_{\mathbf{X}'})$ is also aligned or complete or satisfies the join-condition.

Proof: If $\mathbf{C}_0=\emptyset$, then $(\mathbf{X},\mathcal{C}_{\mathbf{X}})=(\mathbf{X}',\mathcal{C}'_{\mathbf{X}'})$ and the claim trivially follows. Hence, assume that $\mathbf{C}_0\neq 0$.

(i) Let $\mathcal{N}\subseteq\mathcal{C}'_{\mathbf{X}'}$ be a nested chain. For each $\mathbf{C}\in\mathcal{N}$, there is a $\bar{\mathbf{C}}\in\mathcal{C}_{\mathbf{X}}$ with $\bar{\mathbf{C}}=\mathbf{C}\cup\mathbf{C}_0$. Obviously, $\bar{\mathcal{N}}=\{\bar{\mathbf{C}}\mid\mathbf{C}\in\mathcal{N}\}$ is also a nested chain with $\bigcup\bar{\mathcal{N}}=\bigcup\mathcal{N}\cup\mathbf{C}_0\in\mathcal{C}_{\mathbf{X}}$. Hence, $\bigcup\mathcal{N}\in\mathcal{C}'_{\mathbf{X}'}$.

(ii) Let $\mathbf{C}\subseteq\mathbf{X}'$ such that, for all $x,y\in\mathbf{C}$, we have $\emptyset\neq\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\{x,y\})\subseteq\mathbf{C}$. Consider $\bar{\mathbf{C}}=\mathbf{C}\cup\mathbf{C}_0$. We show that $\bar{\mathbf{C}}$ is convex. Since $(\mathbf{X},\mathcal{C}_{\mathbf{X}})$ is complete, it suffices to prove that $\bar{\mathbf{C}}$ is visibility convex. Clearly, we have, for $x,y\in\mathbf{C}$, that $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\})=\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\{x,y\})\cup\mathbf{C}_0\subseteq\bar{\mathbf{C}}$. If $x\in\mathbf{C}$ and $y\in\mathbf{C}_0$, then $\{x,y\}\subseteq\{x\}\cup\mathbf{C}_0\in\mathcal{C}_{\mathbf{X}}$ and, hence, $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\})=\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\{x\})\cup\mathbf{C}_0=\{x\}\cup\mathbf{C}_0\subseteq\bar{\mathbf{C}}$. Finally, if $x,y\in\mathbf{C}_0$, then $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\})=\mathbf{C}_0\subseteq\bar{\mathbf{C}}$. Since in each of these cases $\mathbf{C}_0$ belongs to $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\})$, it is also non-empty. Hence, $\bar{\mathbf{C}}$ is $\mathcal{C}_{\mathbf{X}}$-visibility convex and, by the completeness of $(\mathbf{X},\mathcal{C}_{\mathbf{X}})$ we have $\bar{\mathbf{C}}\in\mathcal{C}_{\mathbf{X}}$. Therefore, $\mathcal{C}'_{\mathbf{X}'}\text{-}hull(\mathbf{C})=\mathcal{C}_{\mathbf{X}}\text{-}hull(\bar{\mathbf{C}})\setminus\mathbf{C}_0=\bar{\mathbf{C}}\setminus\mathbf{C}_0=\mathbf{C}$ and $\mathbf{C}$ is in $\mathcal{C}'_{\mathbf{X}'}$.

(iii) Let x be in $\mathbf{X}'$ and $\mathbf{C}$ in $\mathcal{C}'_{\mathbf{X}'}$. Our first claim is that $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C} \cup \mathbf{C}_0) = \mathcal{C}'_{\mathbf{X}'}$-*join*$(x, \mathbf{C}) \cup \mathbf{C}_0$ if $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C} \cup \mathbf{C}_0)$ is non-empty. So assume $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C} \cup \mathbf{C}_0) \neq \emptyset$. Obviously, we have $\mathcal{C}'_{\mathbf{X}'}$-*join*$(x, \mathbf{C}) \cup \mathbf{C}_0 \subseteq \mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C} \cup \mathbf{C}_0)$ because $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C} \cup \mathbf{C}_0) \neq \emptyset$. To see the reverse inclusion let $y \in \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, c\})$ with $c \in \mathbf{C} \cup \mathbf{C}_0$. If $c \in \mathbf{C}$, then $\emptyset \neq \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, c\}) = \mathcal{C}'_{\mathbf{X}'}$-*hull*$(\{x, c\}) \cup \mathbf{C}_0$ by Lemma 4.5. Since $x \in \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, c\}) \setminus \mathbf{C}_0$, we have $\mathcal{C}'_{\mathbf{X}'}$-*hull*$(\{x, c\}) \neq \emptyset$, for all $c \in \mathbf{C}$ and, thus, $\mathcal{C}'_{\mathbf{X}'}$-*hull*$(\{x, c\}) \subseteq \mathcal{C}'_{\mathbf{X}'}$-*join*$(x, \mathbf{C})$. This implies $y \in \mathcal{C}'_{\mathbf{X}'}$-*join*$(x, \mathbf{C}) \cup \mathbf{C}_0$. If $c \in \mathbf{C}_0$, then $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, c\}) = \{x\} \cup \mathbf{C}_0 \subseteq \mathcal{C}'_{\mathbf{X}'}$-*join*$(x, \mathbf{C}) \cup \mathbf{C}_0$.

Since $\mathbf{C} \in \mathcal{C}'_{\mathbf{X}'}$, we have $\mathbf{C} \cup \mathbf{C}_0 \in \mathcal{C}_{\mathbf{X}}$ and $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C} \cup \mathbf{C}_0)$ is $\mathcal{C}_{\mathbf{X}}$-convex because $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ satisfies the join condition. Therefore, $\mathcal{C}'_{\mathbf{X}'}$-*join*$(x, \mathbf{C}) = \mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C} \cup \mathbf{C}_0) \setminus \mathbf{C}_0$ is $\mathcal{C}'_{\mathbf{X}'}$-convex.

If $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C} \cup \mathbf{C}_0) = \emptyset$, then there exists a $c \in \mathbf{C} \cup \mathbf{C}_0$ with $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, c\}) = \emptyset$. c has to be in $\mathbf{C}$ since $\{x\} \cup \mathbf{C}_0 \subseteq \mathbf{X}$ is a convex set containing $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, c\})$, for $c \in \mathbf{C}_0$. Hence, $\{\bar{\mathbf{C}} \in \mathcal{C}_{\mathbf{X}} \mid \{x, c\} \subseteq \bar{\mathbf{C}}\} = \emptyset$ and, thus, $\{\mathbf{C}' \in \mathcal{C}'_{\mathbf{X}'} \mid \{x, c\} \subseteq \mathbf{C}'\} = \emptyset$. It follows that $\mathcal{C}'_{\mathbf{X}'}$-*hull*$(\{x, c\}) = \emptyset$ and, therefore, $\mathcal{C}'_{\mathbf{X}'}$-*join*$(x, \mathbf{C}) = \emptyset$. So $\mathcal{C}'_{\mathbf{X}'}$-*join*$(x, \mathbf{C})$ is $\mathcal{C}'_{\mathbf{X}'}$-convex in this case, too.

$\square$

The above theorem shows that we do not gain any generality in considering near-simple semi-convex spaces and that it suffices to restrict our attention to simple semi-convex spaces whenever we deal with the Kernel Theorem.

4.2 Convexity of Kernels

The Kernel Theorem gives rise to two natural follow up questions.

1. Can we guarantee the convexity of kernels under weaker conditions than those necessary for the Kernel Theorem ?

2. Is there a relationship between kernels and skulls in general semi-convex spaces ?

Let us consider the convexity of kernels first. Algorithms that compute kernels of planar sets often make crucial use of the fact that they are convex. Therefore, it is desirable to have a characterization of those semi-convex spaces for which this is true. Unfortunately, there is a large gap between the sufficient and necessary conditions for the convexity of kernels as stated in the following theorems.

Theorem 4.7 *Let $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ be a complete semi-convex space. If $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ satisfies the join condition, then $\mathcal{C}_{\mathbf{X}}$-kernel$(\mathbf{Y})$ is convex, for all $\mathbf{Y} \subseteq \mathbf{X}$.*

Proof: Let $\mathbf{Y} \subseteq \mathbf{X}$ and, w.l.o.g. we assume that $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y}) \neq \emptyset$. Consider $x, y \in \mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y})$. Since x and y can see each other, we have $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, y\}) \neq \emptyset$ and $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x, y\}) \subseteq \mathbf{Y}$. Now take any $z \in \mathbf{Y}$. We want to show that any point w

in $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y\})$ can see z. This implies that w belongs to $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y})$ and, hence, $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y})$ is visibility convex.

Let $\mathbf{Q}$ be defined as $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathcal{C}_{\mathbf{X}}$-*hull*$(\{y,z\}))$. We have that $\mathbf{Q} \subseteq \mathbf{Y}$ and that $\mathbf{Q}$ is non-empty and since $\emptyset \neq \mathcal{C}_{\mathbf{X}}$-*hull*$(\{y,z\}) \subseteq \mathbf{Y}$ and x can see all points in $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{y,z\})$. Now let w be a point in $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y\})$. Since, $\mathbf{Q} = \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y,z\}) \supseteq \mathcal{C}_{\mathbf{X}}$-*join*$(z, \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y\}))$ by the join-property, we have that $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{w,z\}) \subseteq \mathbf{Y}$. Further, $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{w,z\}) \neq \emptyset$ because $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{w,z\}) \subseteq \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y,z\}) \neq \emptyset$. But this implies that $w \in \mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y})$, since z was arbitrary. Hence, $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y})$ is visibility convex and by the completeness of $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ convex.

□

The assumption of a complete space cannot be relaxed as the next theorem shows.

Corollary 4.8 *If* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is a semi-convex space such that, for all* $\mathbf{Y} \subseteq \mathbf{X}$, $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y})$ *is convex, then* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is complete.*

Proof: Let $\mathbf{C} \subseteq \mathbf{X}$ be a visibility convex set. Since we then have that $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{C}) = \mathbf{C}$, the convexity of kernels immediately implies that $\mathbf{C} \in \mathcal{C}_{\mathbf{X}}$. □

The next lemma gives additional conditions which imply the join-condition if kernels are convex.

Lemma 4.9 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space such that, for all* $\mathbf{Y} \subseteq \mathbf{X}$, $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y})$ *is convex; if, for any three points* x, y, *and* $z \in \mathbf{X}$, *we have*

$$\begin{aligned} \mathcal{C}_{\mathbf{X}}\text{-}join(x, \mathcal{C}_{\mathbf{X}}\text{-}hull(\{y,z\})) &= \mathcal{C}_{\mathbf{X}}\text{-}join(y, \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,z\})) \qquad (1) \\ &= \mathcal{C}_{\mathbf{X}}\text{-}join(z, \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\})), \qquad (2) \end{aligned}$$

then $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *satisfies the join condition.*

Proof: By Corollary 4.8 and Theorem 3.7 we only have to show that, for any three points x, y, and $z \in \mathbf{X}$, $\mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathcal{C}_{\mathbf{X}}$-*hull*$(\{y,z\}))$ is convex. Assume w.l.o.g. that $\mathbf{Q} = \mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathcal{C}_{\mathbf{X}}$-*hull*$(\{y,z\})) \neq \emptyset$. Clearly, $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Q}) \supseteq \{x,y,z\}$. Since $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Q})$ is convex, we have $\mathbf{Q} \supseteq \mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Q}) \supseteq \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y,z\}) \supseteq \mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathcal{C}_{\mathbf{X}}$-*hull*$(\{y,z\})) = \mathbf{Q}$. Hence, $\mathbf{Q}$ is convex. □

We will now look at a different conditions that imply convexity of kernels. Hence, we start with a new definition.

Definition 4.2 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space. We say* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *satisfies the* star visibility condition *if, for all points* $x \in \mathbf{X}$, *convex sets* $\mathbf{C} \in \mathcal{C}_{\mathbf{X}}$, *and points* $y \in \mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C})$, *we have that if* p *is a point in* $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y\})$, *then* $\mathcal{C}_{\mathbf{X}}$-*star*$(p, \mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C})) \supseteq \mathcal{C}_{\mathbf{X}}$-*star*$(x, \mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C})) \cap \mathcal{C}_{\mathbf{X}}$-*star*$(y, \mathcal{C}_{\mathbf{X}}$-*join*$(x, \mathbf{C}))$.

Lemma 4.10 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a complete semi-convex space; if the star visibility condition holds, then, for all* $\mathbf{Y} \subseteq \mathbf{X}$, *we have* $\mathcal{C}_{\mathbf{X}}$-*kernel*$(\mathbf{Y})$ *is convex.*

Proof: W.l.o.g. let $\mathbf{Y} \subseteq \mathbf{X}$ and $x, y \in \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y})$. We want to show $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y})$. Since x and y belong to the $\mathcal{C}_{\mathbf{X}}\text{-}kernel$ of $\mathbf{Y}$, we obviously have $\emptyset \neq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \mathbf{Y}$. Now let $p \in \mathbf{Y}$ be arbitrary. Since $y \;\; \mathcal{C}_{\mathbf{Y}}\text{-}sees \;\; p$, we have also $\emptyset \neq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{y,p\}) \subseteq \mathbf{Y}$. Furthermore, since x is in the kernel, $\mathbf{Q} = \mathcal{C}_{\mathbf{X}}\text{-}join(x, \mathcal{C}_{\mathbf{X}}\text{-}hull(\{y,p\}))$ is non-empty and contained in $\mathbf{Y}$. Now, if $z \in \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\})$, then $\mathcal{C}_{\mathbf{X}}\text{-}star(z,\mathbf{Q}) \supset \mathcal{C}_{\mathbf{X}}\text{-}star(x,\mathbf{Q}) \cap \mathcal{C}_{\mathbf{X}}\text{-}star(y,\mathbf{Q}) \supset \{p\}$. Hence, $z \in \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y})$. □

With these two results we can actually see that the join condition and the star visibility property are equivalent in complete spaces.

Lemma 4.11 *Let $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ be a complete semi-convex space. $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ satisfies the star visibility condition if and only if $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ satisfies the join condition.*

Proof: **if**. Since the star visibility-property implies the convexity of kernels as shown in Lemma 4.10, we only have to show that, for any three points $x, y, z \in \mathbf{X}$, we have $\mathbf{Q} = \mathcal{C}_{\mathbf{X}}\text{-}join(x, \mathcal{C}_{\mathbf{X}}\text{-}hull(\{y,z\}))$ equals $\mathcal{C}_{\mathbf{X}}\text{-}join(y, \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,z\}))$ by Lemma 4.9 . Hence, let $p \in \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,z\})$. We want to show that $p \;\; \mathcal{C}_{\mathbf{Q}}\text{-}sees \;\; y$. Since $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y,x\}) \subseteq \mathbf{Q}$ and $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{y,z\}) \subseteq \mathbf{Q}$, we have $\mathcal{C}_{\mathbf{X}}\text{-}star(p,\mathbf{Q}) \supseteq \mathcal{C}_{\mathbf{X}}\text{-}star(x,\mathbf{Q}) \cap \mathcal{C}_{\mathbf{X}}\text{-}star(z,\mathbf{Q}) \supseteq \{y\}$ and $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{p,y\}) \subseteq \mathbf{Q}$. Hence, $\mathcal{C}_{\mathbf{X}}\text{-}join(y, \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,z\})) \subseteq \mathbf{Q}$. The same argument works for the reverse inclusion as well; so, the claimed equality holds.
only if. If $\mathbf{Q} = \mathcal{C}_{\mathbf{X}}\text{-}join(x, \mathbf{C})$ is convex, then, for any point $y \in \mathbf{Q}$, we have that $\mathcal{C}_{\mathbf{X}}\text{-}star(y,\mathbf{Q}) = \mathbf{Q}$ and the star visibility property trivially holds. □

4.3 An Extension to the Kernel Theorem

We turn now to the second question: What is the relationship between $\mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$ and $\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y})$, for $\mathbf{Y} \subseteq \mathbf{X}$, in an arbitrary semi-convex space? It is obvious that we have to require the existence of skulls in order to be able to make a statement that relates skulls and kernels. Taking this condition into account the next theorem gives the most general connection between skulls and kernels possible.

But before we can state and prove a generalized version of the Kernel Theorem, we need the notion of a skull cover.

Definition 4.3 *Given a semi-convex space $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ and a set $\mathbf{Y} \subseteq \mathbf{X}$, we say that a subset $\mathcal{S}$ of $\mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$ is a* skull cover *of $\mathbf{Y}$ if $\bigcup \mathcal{S} = \mathbf{Y}$. We define the collection of skull covers of $\mathbf{Y}$ as the set $\{\mathcal{S} \subseteq \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y}) \mid \mathcal{S} \text{ is a skull cover of } \mathbf{Y}\}$ and denote it by $\Sigma(\mathbf{Y})$.*

The theorem can now be stated as follows.

Theorem 4.12 (Cover Kernel Theorem) *If $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is a semi-convex space, then we have, for all $\mathbf{Y} \subseteq \mathbf{X}$, that*

$$\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y}) = \bigcup_{\mathcal{S} \in \Sigma(\mathbf{Y})} \left(\bigcap \mathcal{S}\right)$$

if and only if, for all convex subsets $\mathbf{C}$ of $\mathbf{Y}$, there is a $\mathbf{S} \in \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$ with $\mathbf{C} \subseteq \mathbf{S}$.

Proof: Let $\mathbf{Y}$ be a non-empty subset of $\mathbf{X}$.
if. Let $K = \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y})$ and $I = \bigcup_{\mathcal{S}\in\Sigma(\mathbf{Y})}(\bigcap \mathcal{S})$. We first prove that $K \subseteq I$ and then prove that $I \subseteq K$.

$K \subseteq I$. W.l.o.g. assume that $K \neq \emptyset$. Let $x \in K$ and define $\mathcal{S}_x = \{\mathbf{S} \in \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y}) \mid x \in \mathbf{S}\}$. Since x can see all points in $\mathbf{Y}$, we have that, for all $y \in \mathbf{Y}$, $\emptyset \neq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \mathbf{Y}$; that is, there exists an $\mathbf{S} \in \mathcal{S}_x$ such that $\{x,y\} \subseteq \mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \mathbf{S} \in \mathcal{S}_x$ since the existence of skulls for any convex subset $\mathbf{C}$ of $\mathbf{Y}$ is guaranteed. Hence, $\mathcal{S}_x$ is a cover of $\mathbf{Y}$ with $x \in \bigcap \mathcal{S}_x \subseteq I$.

$I \subseteq K$. Let $x \in I$; then, there is an $\mathcal{S} \in \Sigma(\mathbf{Y})$ such that $x \in \bigcap \mathcal{S}$. Let y be an arbitrary point in $\mathbf{Y}$; then, since $\mathcal{S}$ is a cover of $\mathbf{Y}$, there is a skull $\mathbf{S} \in \mathcal{S}$ such that $y \in \mathbf{S}$. Because $x \in \bigcap \mathcal{S}$, we know that $x \in \mathbf{S}$, so $\{x,y\} \subseteq \mathbf{S}$; since $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \neq \emptyset$ and $\mathcal{C}_{\mathbf{X}}\text{-}hull(\{x,y\}) \subseteq \mathbf{S} \subseteq \mathbf{Y}$, we have $x\ \mathcal{C}_{\mathbf{Y}}\text{-}sees\ y$ and $x \in \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y})$.

only if. Let $\mathbf{Y} \subseteq \mathbf{X}$ and $\mathbf{C}$ a non-empty convex subset of $\mathbf{Y}$. We have to show that there is a skull of $\mathbf{Y}$ that contains $\mathbf{C}$. Consider the family $\mathcal{S}_{\mathbf{C}} = \{\mathbf{C}' \in \mathcal{C}_{\mathbf{X}} \mid \mathbf{C} \subseteq \mathbf{C}' \subseteq \mathbf{Y}\}$ and let $\mathbf{Q}$ be the set $\bigcup \mathcal{S}_{\mathbf{C}}$. $\mathbf{Q}$ is non-empty since $\mathbf{C} \in \mathcal{S}_{\mathbf{C}}$. Clearly, $\mathbf{C} \subseteq \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Q})$ and since $\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Q}) = \bigcup_{\mathcal{S}\in\Sigma(\mathbf{Q})}(\bigcap \mathcal{S})$, there has to be at least one skull $\mathbf{S}$ contained in $\mathbf{Q}$ with $\mathbf{C} \subseteq \mathbf{S}$. Since all convex of $\mathbf{Y}$ sets that contain $\mathbf{C}$ are also contained in $\mathbf{Q}$, we have $\mathbf{S} \in \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y})$. □

The only if part of the above proof shows why it is impossible to make any statement about the relationship between skulls and kernels without assuming that, for all $\mathbf{Y} \subseteq \mathbf{X}$ and all convex subsets $\mathbf{C}$ of $\mathbf{Y}$, there exists a skull $\mathbf{S}$ that contains $\mathbf{C}$. For, let $\mathbf{Y}$ and $\mathbf{C}$ be chosen such that there is no skull $\mathbf{S}$ with $\mathbf{C} \subseteq \mathbf{S} \subseteq \mathbf{Y}$. Consider, as in the proof, the family $\mathcal{S}_{\mathbf{C}}$ of all convex subsets of $\mathbf{Y}$ that contain $\mathbf{C}$ and let $\mathbf{Q}$ be their union. Since there is no skull of $\mathbf{C}$ in $\mathbf{Q}$ and the kernel of $\mathbf{Q}$ contains $\mathbf{C}$, any assertion of the form $\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Q}) = F(\mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Q}))$ where F is some function must be wrong.

Clearly, we can restrict ourselves to *minimal* skull covers in the above theorem. This gives us a slightly stronger version of the first Kernel Theorem by the following argument. If $\mathcal{S}_0$ is some minimal cover of $\mathbf{Y}$, then

$$\begin{array}{rcll} \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y}) & = & \bigcap \mathcal{C}_{\mathbf{X}}\text{-}skulls(\mathbf{Y}) & \text{by the Kernel Theorem} \\ & \subseteq & \bigcap \mathcal{S}_0 & \\ & \subseteq & \bigcup_{\mathcal{S}\in\Sigma(\mathbf{Y})}(\bigcap \mathcal{S}) & \\ & = & \mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y}) & \end{array}$$

and, hence, $\mathcal{C}_{\mathbf{X}}\text{-}kernel(\mathbf{Y}) = \bigcap \mathcal{S}$, for *any* minimal cover $\mathcal{S}$ of $\mathbf{Y}$.

4.4 Subspaces of Semi-convex Spaces

One of the main reasons we introduced semi-convex spaces was that they are closed under visibility subspace formation. As it turns out most of the properties investigated here also carry over to subspaces. But before we show this we need a technical lemma.

Lemma 4.13 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space,* $\mathbf{Y} \subseteq \mathbf{X}$ *and* $\mathcal{C}_{\mathbf{Y}}$ *defined as before. If, for all* $x, y \in \mathbf{Y}$, $\mathcal{C}_{\mathbf{Y}}$*-hull*$(\{x,y\}) \neq \emptyset$*, then* $\mathcal{C}_{\mathbf{Y}}$*-hull*$(\{x,y\}) = \mathcal{C}_{\mathbf{X}}$*-hull*$(\{x,y\})$.

Proof: $\mathcal{C}_{\mathbf{Y}}$-*hull*$(\{x,y\}) \supseteq \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y\})$ since $\mathcal{C}_{\mathbf{Y}}$-*hull*$(\{x,y\}) \in \mathcal{C}_{\mathbf{X}}$ and $\mathcal{C}_{\mathbf{Y}}$-*hull*$(\{x,y\}) \neq \emptyset$. On the other hand, since $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y\}) \subseteq \mathcal{C}_{\mathbf{Y}}$-*hull*$(\{x,y\}) \subseteq \mathbf{Y}$ and $\mathcal{C}_{\mathbf{Y}}$-*hull*$(\{x,y\}) \neq \emptyset$ by Theorem 3.1 Property (v), we have $\{x,y\} \subseteq \mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y\}) \in \mathcal{C}_{\mathbf{Y}}$ and, hence, $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{x,y\}) \supseteq \mathcal{C}_{\mathbf{Y}}$-*hull*$(\{x,y\})$ as $\mathcal{C}_{\mathbf{Y}}$-*hull*$(\{x,y\})$ is the smallest $\mathcal{C}_{\mathbf{X}}$-convex set in $\mathbf{Y}$ that contains x and y. □

We are now in a position to state the main theorem about subspaces.

Theorem 4.14 *Let* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *be a semi-convex space,* $\mathbf{Y} \subseteq \mathbf{X}$*. If* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *is*

(i) simple or

(ii) aligned or

(iii) complete or

(iv) satisfies the join-property,

then $(\mathbf{Y}, \mathcal{C}_{\mathbf{Y}})$ *is also simple or aligned or complete or satisfies the join-property.*

Proof: We prove the properties in the sequence as they appear in the claim.

(i) Let $y \in \mathbf{Y}$; then, $\{y\} \in \mathcal{C}_{\mathbf{X}}$ since $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is simple and $\{y\} \subseteq \mathbf{Y}$. Thus, $\{y\} \in \mathcal{C}_{\mathbf{Y}}$ and $(\mathbf{Y}, \mathcal{C}_{\mathbf{Y}})$ is simple.

(ii) Let $\mathcal{N}$ be a nested chain in $\mathcal{C}_{\mathbf{Y}}$. Then, for every $\mathbf{C} \in \mathcal{N}$, we have $\mathbf{C} \subseteq \mathbf{Y}$ and, hence, $\bigcup \mathcal{N} \subseteq \mathbf{Y}$. Since $\mathcal{N} \subseteq \mathcal{C}_{\mathbf{X}}$, we have that by the alignedness of $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ that $\bigcup \mathcal{N} \in \mathcal{C}_{\mathbf{X}}$ and, therefore, $\bigcup \mathcal{N} \in \mathcal{C}_{\mathbf{Y}}$.

(iii) Obvious, since the completeness of a semi-convex space only depends on the properties of the $\mathcal{C}_{\mathbf{X}}$-hull of two points.

(iv) Let $y \in \mathbf{Y}$ and $\mathbf{C} \in \mathcal{C}_{\mathbf{Y}}$ such that, for all $c \in \mathbf{C}$, we have $\mathcal{C}_{\mathbf{Y}}$-*hull*$(\{y,c\}) \neq \emptyset$. Hence, $\mathcal{C}_{\mathbf{X}}$-*hull*$(\{y,c\}) \neq \emptyset$ and $\mathcal{C}_{\mathbf{X}}$-$join(y, \mathbf{C})$ is non-empty. Since $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ satisfies the join-property, $\mathcal{C}_{\mathbf{X}}$-$join(y, \mathbf{C}) \in \mathcal{C}_{\mathbf{X}}$. Further, by the above lemma we have that $\mathcal{C}_{\mathbf{Y}}$-$join(y, \mathbf{C}) = \mathcal{C}_{\mathbf{X}}$-$join(y, \mathbf{C})$ and $\mathcal{C}_{\mathbf{Y}}$-$join(y, \mathbf{C}) \subseteq \mathbf{Y}$. Hence, $\mathcal{C}_{\mathbf{Y}}$-$join(y, \mathbf{C}) \in \mathcal{C}_{\mathbf{Y}}$ and $(\mathbf{Y}, \mathcal{C}_{\mathbf{Y}})$ satisfies the join-property.

□

Please note that near-simplicity does not carry over to subspaces since, for instance, we may get points with an empty hull.

5 Generating Semi-convex Spaces

This section is concerned with different possibilities to define semi-convex spaces. If the family of convex sets is not given by enumeration which is, of course, only possible if we consider finite semi-convex spaces, then the convexity of a set depends on a specific condition that has to be satisfied. The most commonly used condition is to require the "hull" of any two points to be contained in the considered set. As an example consider the usual convexity in the plane which is based on line segments as the hull of two points.

5.1 The Hull Function

So suppose we are given a function $hull : \mathbf{X} \times \mathbf{X} \longrightarrow 2^{\mathbf{X}}$ and specify our family of convex sets $\mathcal{C}_{\mathbf{X}}$ by $\mathcal{C}_{\mathbf{X}} = \{\mathbf{C} \subseteq \mathbf{X} \mid$ for all x and $y \in \mathbf{C}$: $hull(x, y) \neq \emptyset$, $hull(y, x) \neq \emptyset$, and $\mathbf{C}$ contains $hull(x, y) \cup hull(y, x)\}$. We show that this is, of course, the definition of a complete semi-convex space.

Theorem 5.1 *If hull be a function from* $\mathbf{X} \times \mathbf{X}$ *into* $2^{\mathbf{X}}$*, then* $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ *as defined above is a complete semi-convex space.*

Proof: Note that the emptyset satisfies the above definition of convex set. Therefore, we only have to show the closure under intersection. Let $\mathcal{C} \subseteq \mathcal{C}_{\mathbf{X}}$ be a family of convex sets and $x, y \in \bigcap \mathcal{C}$. Since $hull(x, y) \subseteq \mathbf{C}$, for all $\mathbf{C} \in \mathcal{C}$, have $hull(x, y) \subseteq \bigcap \mathcal{C}$. The same holds true for $hull(y, x)$. Hence, $\bigcap \mathcal{C}$ satisfies the above definition and is $\mathcal{C}_{\mathbf{X}}$-convex. $(\mathbf{X}, \mathcal{C}_{\mathbf{X}})$ is complete because if $\mathbf{C}$ is a subset of $\mathbf{X}$ such that, for all $x, y \in \mathbf{C}$, we have $\mathcal{C}_{\mathbf{X}}$-$hull(\{x, y\}) \subseteq \mathbf{C}$, then, clearly, $hull(x, y) \cup hull(y, x) \subseteq \mathcal{C}_{\mathbf{X}}$-$hull(\{x, y\})$ by the definition of $\mathcal{C}_{\mathbf{X}}$-$hull(\{x, y\})$ and $\mathbf{C}$ is convex. □

It is an interesting question when we actually have that $hull(x, y)$ equals $\mathcal{C}_{\mathbf{X}}$-$hull(\{x, y\})$, for all x, y. We give an answer in the following lemma.

Theorem 5.2 *Let hull and* $\mathcal{C}_{\mathbf{X}}$ *be defined as above. We have, for all* $x, y \in \mathbf{X}$*, that* $hull(x, y) = \mathcal{C}_{\mathbf{X}}$-$hull(\{x, y\})$ *if and only if hull is symmetric and if* $hull(x, y) \neq \emptyset$*, then, for all* $p, q \in hull(x, y)$*,* $\{p, q\} \subseteq hull(p, q) \subseteq hull(x, y)$*.*

Proof: The necessity of the two conditions follows immediately from the properties of the hull-operator (see Theorem 3.1). So we only have to show the sufficiency. So let $hull$ be a symmetric function that satisfies $hull(p, q) \subseteq hull(x, y)$, for all $p, q \in hull(x, y)$. By definition we have $hull(x, y) \subseteq \mathcal{C}_{\mathbf{X}}$-$hull(\{x, y\})$ so we only have to show the reverse inclusion. But this is obvious since the condition is just a reformulation of the definition of a visibility convex set. Hence, $hull(x, y)$ is convex. If $hull(x, y) \neq \emptyset$, then the inclusion follows since $\mathcal{C}_{\mathbf{X}}$-$hull(\{x, y\})$ is the smallest convex set that contains $\{x, y\}$ and, thus, $hull(x, y) \supseteq \mathcal{C}_{\mathbf{X}}$-$hull(\{x, y\})$. If $hull(x, y) = \emptyset$, then there is no convex set that contains x and y since we defined a convex set $\mathbf{C}$ to have $hull(x', y') \neq \emptyset$, for all $x', y' \in \mathbf{C}$. □

5.2 Distance Functions

Although the method outlined above of defining semi-convex spaces works satisfactorily in most cases since only the hull of two points has to be specified, it is sometimes desirable to further simplify the definition or to add more structure to the hull function if possible. One way to proceed in these cases is to use *distance functions*. A *distance function* d is a mapping $d : \mathbf{X} \times \mathbf{X} \longrightarrow I\!R_+$ that satisfies the following two properties, for all $x, y \in \mathbf{X}$.

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$ (positive definiteness)
2. $d(x, y) = d(y, x)$ (symmetry)

We say that a point z *lies between* the points x and y w.r.t. d if $d(x, z) + d(z, y) = d(x, y)$. The definition of the hull of two points x and y can now be based on the points that lie between x and y w.r.t. d, i.e., the hull of x and y can be defined as the set of all points z in $\mathbf{X}$ that satisfy $d(x, z) + d(z, y) = d(x, y)$. We will not further investigate the consequences of this definition. Instead we take a different approach. We want to stress the importance of connectedness of the convex hull since it is our aim to model the notion of visibility or reachability with semi-convex spaces.

So our definition of the convex hull of two points is based on *d-straight paths*. A *d*-straight path $\mathcal{P}$ from p to q is a shortest path in $\mathbf{X}$ from p to q that consists only of points that lie between p and q.

Definition 5.1 *Let* $\mathbf{X}$ *be some set, d a distance function, and x, y be two points in* $\mathbf{X}$*. We define the $\mathcal{C}_d$-hull* of x and y *as* $\bigcup\{\mathcal{P} \mid \mathcal{P}$ *is a d-straight path connecting x and y*$\}$ *and denote it by $\mathcal{C}_d$-hull*$(\{x, y\})$.

We denote the semi-convex space induced by d by $(\mathbf{X}, \mathcal{C}_d)$. It is not obvious that the above definition conforms with our concept of convex hull and it is not to difficult to find counterexamples of distance functions for which $\mathcal{C}_d$-*hull* is not a hull function. But if we restrict the spaces and distance functions appropriately, it can be shown that they satisfy the conditions of Theorem 5.2. As an example consider the distance function L_2. The $\mathcal{C}_{L_2}$-*hull*$(\{p, q\})$ is just the line segment between p and q and which again leads to the usual definition of convexity.

5.3 Metrics in the Plane

A very interesting special case of distance functions are *metrics* which have received a lot of attention in both computer science and mathematics. A metric d additionally satisfies the following property

3. $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in \mathbf{X}$. (triangle inequality)

In the following we will show that the semi-convex spaces which are induced by metrics that have certain additional properties satisfy the join-condition. Since the semi-convex spaces induced by distance functions are always simple and complete, this implies that these semi-convex spaces satisfy the Kernel theorem. From now on throughout the rest of this paper we will assume that $\mathbf{X}$ is a simple domain. Recall that a simple domain is a subset of the plane that is homeomorphic to the closure of the unit ball.

So let $\mathbf{X}$ be a simple domain and d metric in it. In order to show that d satisfies the join property several conditions have to be met. First we require that d fits reasonably well into the topology of $\mathbb{E}^2$. We say d is *topological* if d satisfies the following three conditions

1. d induces the Euclidean topology in $\mathbf{X}$.
2. The d-circles are bounded w.r.t. the Euclidean metric.
3. There is at least one d-straight path in $\mathbb{E}^2$ between any two points in $\mathbf{X}$.

The second condition we impose requires that we can prolong d-straight paths sufficiently. We say d is *extensible* if d satisfies the following two conditions.

(i) For any d-straight path $\mathcal{P}$, there is a d-straight extension of $\mathcal{P}$ that partitions $\mathbf{X}$ into at least two components and

(ii) all points contained in a d-triangle—that is a Jordan curve which consists of three d-straight paths—can be connected to the corner points of the d-triangle by a d-straight path in $\mathbf{X}$.

Finally, we need a third property to ensure that d satisfies the join condition.

> If p, q, r, and s are four points in $\mathbf{X}$ such that q is in $\mathcal{C}_d$-$hull(\{p,r\})$ and $r \in \mathcal{C}_d$-$hull(\{q,s\})$, then $\mathcal{C}_d$-$hull(\{p,r\})$ is contained in $\mathcal{C}_d$-$join(q, \mathcal{C}_d$-$hull(\{p,s\}))$.

A metric that satisfies the above condition is called *stiff.* Note that any metric that induces a semi-convex space which satisfies the join condition is necessarily stiff since $\mathcal{C}_d$-$join(q, \mathcal{C}_d$-$hull(\{p,s\}))$ is d-convex and p and r are contained in $\mathcal{C}_d$-$join(q, \mathcal{C}_d$-$hull(\{p,s\}))$; hence, we have that $\mathcal{C}_d$-$hull(\{p,r\}) \subseteq \mathcal{C}_d$-$join(q, \mathcal{C}_d$-$hull(\{p,s\}))$. Therefore, it is unavoidable to require stiffness of a metric. The advantage of introducing this new notion is, of course, that it is a lot easier to show that a metric is stiff than to prove that it satisfies the join condition directly.

We are now able to state the following theorem about simple domains which is needed in later sections.

Theorem 5.3 *If $\mathbf{X}$ is a simple domain and d is a topological, extensible, and stiff metric in $\mathbf{X}$, then $(\mathbf{X}, \mathcal{C}_d)$ satisfies the join-property.*

Proof: The proof of this theorem is rather long though not mathematically deep. Because of its length it can be found in Section . We just illustrate the main proof idea. Since the considered semi-convex space is complete, it suffices to prove the join-property for three points.

So consider three points $x, y, z \in \mathbf{X}$. Let $\mathbf{Q}$ be defined as $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$. Consider two points $p, q \in \mathbf{Q}$ and a point $r \in \mathcal{C}_d\text{-}hull(\{p, q\})$. We have to show that r belongs to $\mathbf{Q}$. The convexity of $\mathbf{Q}$ then follows by the completeness of $(\mathbf{X}, \mathcal{C}_d)$.

Let y_p and z_q be two points in $\mathcal{C}_d\text{-}hull(\{y, z\})$ such that $p \in \mathcal{C}_d\text{-}hull(\{x, y_p\})$ and $q \in \mathcal{C}_d\text{-}hull(\{x, z_q\})$. It suffices to prove that $\mathcal{C}_d\text{-}hull(\{p, q\}) \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y_p, z_q\}))$. W.l.o.g. suppose that $y = y_q$ and $z = z_q$. Let $\mathcal{P}_{pq}$ be some d-straight path in $\mathcal{C}_d\text{-}hull(\{p, q\})$. Further, let $\mathcal{P}_{xy}$, $\mathcal{P}_{xz}$, and $\mathcal{P}_{yz}$ be d-straight paths from x to y, x to z, and y to z.

In order to give an impression of the proof idea we assume that $\mathcal{P} = \mathcal{P}_{xy} * \mathcal{P}_{yz} * \mathcal{P}_{zx}$ is a Jordan-curve and that $\mathcal{P}_{pq}$ is contained in the interior of $\mathcal{P}$. If we pick some point p' on $\mathcal{P}_{pq}$, we can connect it to x with a d-straight path $\mathcal{P}_{xp'}$. Since d is extensible, the extension $\mathcal{P}'$ of $\mathcal{P}_{xp'}$ must intersect $\mathcal{P}$. If $\mathcal{P}'$ intersects $\mathcal{P}$ in $\mathcal{P}_{xy}$ or $\mathcal{P}_{yz}$, then it can be seen that p' belongs to $\mathcal{C}_d\text{-}hull(\{x, y\})$ or $\mathcal{C}_d\text{-}hull(\{x, z\})$.

If it intersects $\mathcal{C}_d\text{-}hull(\{y, z\})$, then $p' \in \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$. Hence, $\mathcal{P}_{pq}$ belongs to $\mathbf{Q}$ and this implies that $\mathcal{C}_d\text{-}hull(\{p, q\}) \subseteq \mathbf{Q}$. □

6 Examples

6.1 Real Vector Spaces

As we mentioned earlier, visibility has been studied mostly in the framework of real vector spaces. In the following we will show that the structure of the family $\mathcal{C}_\mathbf{V}$ of convex sets in a real vector space $\mathbf{V}$ is very well behaved. In fact, $(\mathbf{V}, \mathcal{C}_\mathbf{V})$ satisfies all the properties that we have considered in the previous section. Hence, for instance, the Kernel Theorem holds. In [Tor67] a direct proof that real vector spaces satisfy the skull theorem can be found.

Theorem 6.1 *Let $\mathbf{V}$ be a real vector space with the normal definition of convexity, that is, $\mathbf{C} \subseteq \mathbf{V}$ is convex if $\alpha p + (1-\alpha)q \in \mathbf{C}$, for all $p, q \in \mathbf{C}$ and $\alpha \in [0, 1]$. Let $\mathcal{C}_\mathbf{V} = \{\mathbf{C} \subseteq \mathbf{V} \mid \mathbf{C}$ is convex $\}$.*

Then, $(\mathbf{V}, \mathcal{C}_\mathbf{V})$ is a complete, simple semi-convex space that satisfies the join condition.

Proof: The completeness of $(\mathbf{V}, \mathcal{C}_\mathbf{V})$ is an immediate consequence of the definition of a convex set since $\mathcal{C}_\mathbf{V}\text{-}hull(\{p, q\})$ obviously equals $\{x \in \mathbf{V} \mid \text{there is an } \alpha \in [0, 1] \text{ with } x = \alpha p + (1-\alpha)q\}$ and $\mathbf{C} \subseteq \mathbf{V}$ is convex if and only if $\mathcal{C}_\mathbf{V}\text{-}hull(\{p, q\}) \subseteq \mathbf{C}$, for all $p, q \in \mathbf{C}$.

The simplicity of $(\mathbf{V}, \mathcal{C}_\mathbf{V})$ is equally easy to see. Note that simplicity implies Conditions (iii) and (iv) of the Kernel Theorem.

We will use Theorem 3.7 to show that $(\mathbf{V}, \mathcal{C}_\mathbf{V})$ satisfies the join condition.

Let p, q, and $r \in \mathbf{V}$ and consider $\mathbf{Q} = \mathcal{C}_\mathbf{V}\text{-}join(p, \mathcal{C}_\mathbf{V}\text{-}hull(\{q, r\}))$. Since $\mathbf{V}$ is a vector space we can assume without loss of generality that $p = \mathbf{0}$, where $\mathbf{0}$ is the origin of $\mathbf{V}$. Take two arbitrary points $s, t \in \mathbf{Q}$. If we can show that s and t can see each other in $\mathbf{Q}$, we can conclude by the completeness of $(\mathbf{V}, \mathcal{C}_\mathbf{V})$ that $\mathbf{Q}$ is convex. The definition of $\mathbf{Q}$ ensures that there are points $q_s, q_t \in \mathcal{C}_\mathbf{V}\text{-}hull(\{q, r\})$ such that $s \in \mathcal{C}_\mathbf{V}\text{-}hull(\{p, q_s\})$ and

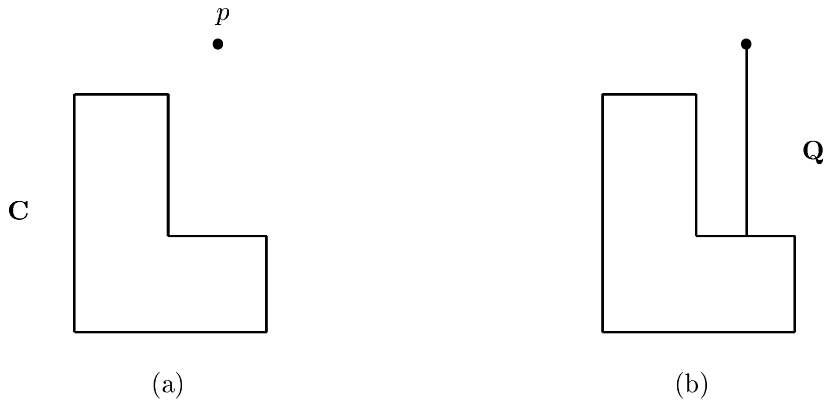


Figure 1: *The join of an* $\{0°, 90°\}$*-convex set and a point.*

$t \in \mathcal{C}_{\mathbf{V}}\text{-}hull(\{p, q_t\})$. In particular, let $\alpha_s \in [0, 1]$ with $s = \alpha_s q_s + (1 - \alpha_s)p = \alpha_s q_s$ and let α_t be defined in the analogous way. Since $\mathbf{C} = \mathcal{C}_{\mathbf{V}}\text{-}join(p, \mathcal{C}_{\mathbf{V}}\text{-}hull(\{q_s, q_t\})) \subseteq \mathbf{Q}$ it suffices to show that s sees t in $\mathbf{C}$. Now choose any $u = \beta s + (1 - \beta)t \in \mathcal{C}_{\mathbf{V}}\text{-}hull(\{s, t\})$, $\beta \in [0, 1]$. If we define λ to be $\beta\alpha_s/(\beta\alpha_s + (1 - \beta)\alpha_t) \in [0, 1]$ and $\mu = 1/(\beta\alpha_s + (1 - \beta)\alpha_t) \in [0, 1]$, it is easy to see that $u = \mu(\lambda q_s + (1 - \lambda)q_t) + (1 - \mu)p \in \mathcal{C}_{\mathbf{V}}\text{-}join(p, \mathcal{C}_{\mathbf{V}}\text{-}hull(\{q_s, q_t\}))$ and hence, $\mathcal{C}_{\mathbf{V}}\text{-}hull(\{s, t\}) \subseteq \mathbf{C} \subseteq \mathbf{Q}$. □

From our previous results we now conclude that the Kernel Theorem holds in all real vector spaces.

6.2 $\{0°, 90°\}$-Convexity

As an example of a semi-convex space that fails to satisfy the conditions of the Kernel Theorem we study the notion of $\{0°, 90°\}$*-convexity.*

We say that a subset $\mathbf{X}$ of $\mathbb{E}^2$ is $\{0°, 90°\}$*-convex* if its intersection with a horizontal or vertical line is empty, a single point or a line segment. Letting $\mathcal{C}_{\mathbb{E}^2}$ be the set of all $\{0°, 90°\}$-convex sets, $(\mathbb{E}^2, \mathcal{C}_{\mathbb{E}^2})$ is a convexity space; see [RW88]. The semi-convex space $(\mathbb{E}^2, \mathcal{C}_{\mathbb{E}^2})$ is simple and complete. However, it does not satisfy the join condition, so the Kernel Theorem does not hold. We provide a counterexample. Let $\mathbf{C}$ be the $\{0°, 90°\}$-convex polygon in Figure 1a and p be the point shown outside $\mathbf{C}$. Then, $\mathbf{Q} = \mathcal{C}_{\mathbb{E}^2}\text{-}join(p, \mathbf{C})$ is given in Figure 1b, and this is, clearly, not $\{0°, 90°\}$-convex. The Cover Kernel Theorem holds, however, as is seen by examining some of the skulls of $\mathbf{Q}$. $\mathbf{C}$ is itself a skull of $\mathbf{Q}$ as are the subsets of $\mathbf{Q}$ shown in Figure 2. Observe that $\mathbf{C}$ and $\mathbf{S}_2$ as well as $\mathbf{S}_1$ and $\mathbf{S}_2$ cover $\mathbf{Q}$. The union of the intersections $(\mathbf{C} \cap \mathbf{S}_2) \cup (\mathbf{S}_1 \cap \mathbf{S}_2)$ which is shown in Figure 3 is already the kernel of $\mathbf{Q}$.

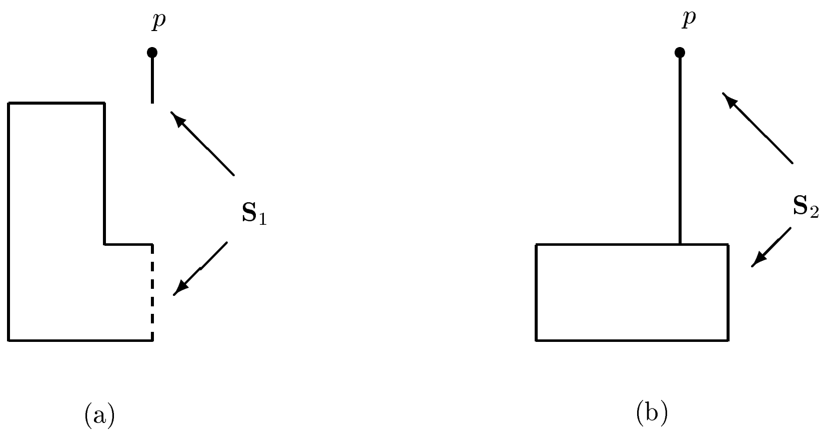


Figure 2: *The* $\{0°, 90°\}$*-skulls of* $\mathbf{Q}$.

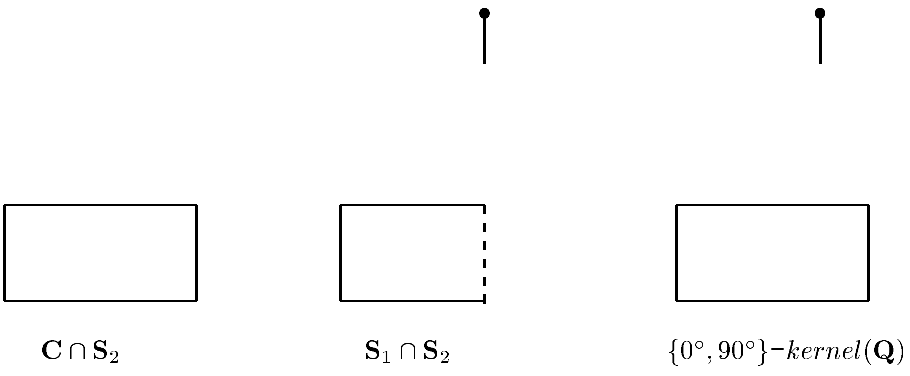


Figure 3: *The* $\{0°, 90°\}$*-kernel of* $\mathbf{Q}$.

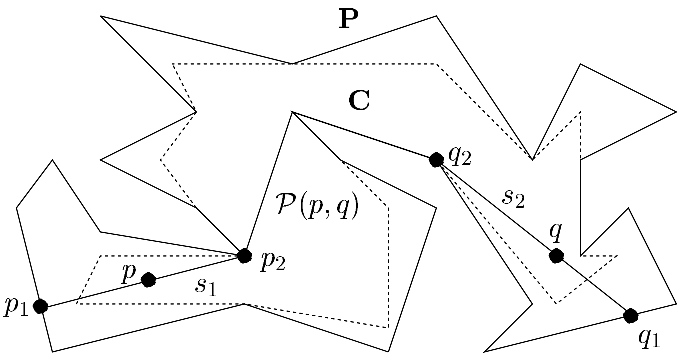


Figure 4: *A g-convex set.*

6.3 Geodesic Convexity

Geodesic convexity provides an example of a semi-convex space that has highly non-linear characteristics but still satisfies all the properties we considered in Section 3.

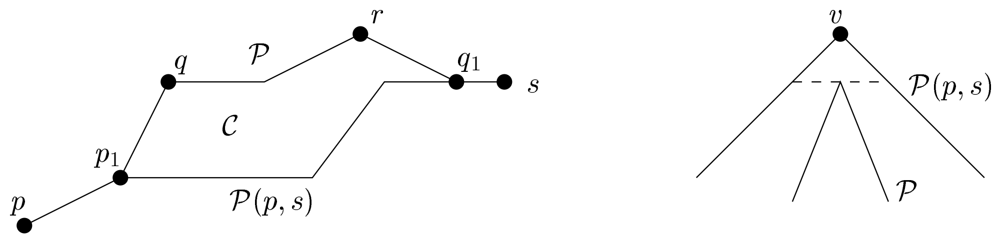
Let $\mathbf{P}$ be a polygon in the plane and for any two points p and q in $\mathbf{P}$ let $\mathcal{P}(p,q)$ denote the shortest path lying wholly in $\mathbf{P}$ that connects p and q. (We use the standard L_2-metric to define the distance between two points.) Note that since $\mathbf{P}$ is simple, there is a *unique* shortest path that connects p and q. We say a subset $\mathbf{C}$ of $\mathbf{P}$ is *geodesically convex* (or *g-convex* for short) if we have that, for every two points p and q in $\mathbf{C}$, $\mathcal{P}(p,q)$ belongs to $\mathbf{C}$. Figure 4 gives an example. Letting $\mathcal{C}_g$ be the set of all g-convex sets in $\mathbf{P}$ $(\mathbf{P}, \mathcal{C}_g)$ is a simple and complete semi-convex space.

Furthermore, $(\mathbf{P}, \mathcal{C}_g)$ satisfies the join condition. To show this we need the following lemma.

Lemma 6.2 *If p, q, r, and s are four points in $\mathbf{P}$ such that q is on $\mathcal{P}(p,r)$ and r is on $\mathcal{P}(q,s)$, then $\mathcal{P}(p,s)$ is the concatenation of $\mathcal{P}(p,r)$ and the part of $\mathcal{P}(q,s)$ from r to s.*

Proof: The proof is by contradiction. Let $\mathcal{P}$ be the curve consisting of the concatenation of $\mathcal{P}(p,r)$ with the part of $\mathcal{P}(q,s)$ from r to s and assume that $\mathcal{P}$ and $\mathcal{P}(p,s)$ differ. Let p_1 be the last point on $\mathcal{P}$ such that the part of $\mathcal{P}$ from p to p_1 equals the part of $\mathcal{P}(p,s)$ from p to p_1 and let q_1 be the first intersection point of $\mathcal{P}$ and $\mathcal{P}(p,s)$ after p_1 on $\mathcal{P}$. For illustration refer to Figure 5a. If we define $\mathcal{C}$ as the curve consisting of the path of $\mathcal{P}$ from p_1 to q_1 concatenated with the part of $\mathcal{P}(p,s)$ from q_1 to p_1, $\mathcal{C}$ is a simple closed curve and, hence, $dom(\mathcal{C})$ is a simple polygon. The interior angles at the vertices of $\mathcal{C}$ are reflex except at p_1 and q_1 since otherwise we can shorten one of $\mathcal{P}$ or $\mathcal{P}(p,s)$ by "shortcutting" a convex vertex as shown in Figure 5b. But if $\mathcal{C}$ consists of n vertices, its sum of interior angles is $(n-2)\cdot 180°$ which contradicts the fact that we have at least $n-2$ reflex vertices. Hence, $\mathcal{P}$ and $\mathcal{P}(p,s)$ cannot differ. □

We now turn to proving that $(\mathbf{P}, \mathcal{C}_g)$ satisfies the join-condition. To this end we, of



(a) (b)

Figure 5: *A convex vertex is shortcut.*

course, employ Theorem 5.3. So we have to show that the inner metric $d_{\mathbf{P}}$ of $\mathbf{P}$, i.e., the induced metric if we measure the distance between two points p and q according to the length of $\mathcal{P}(p,q)$, is topological, extensible, and stiff.

Clearly, $d_{\mathbf{P}}$ is topological since we measure the length of a path with the Euclidean metric. Furthermore, Lemma 6.2 shows that, for any four points p, q, r, and s in $\mathbf{P}$ with $q \in \mathcal{C}_{d_{\mathbf{P}}}$-*hull*$(\{p,r\})$ and $r \in \mathcal{C}_{d_{\mathbf{P}}}$-*hull*$(\{q,s\})$, we have that $\mathcal{C}_{d_{\mathbf{P}}}$-*hull*$(\{p,r\}) = \mathcal{P}(p,r)$ is contained in $\mathcal{P}(p,s) \subseteq \mathcal{C}_{d_{\mathbf{P}}}$-*join*$(q, \mathcal{C}_{d_{\mathbf{P}}}$-*hull*$(\{p,s\}))$ and, hence, $d_{\mathbf{P}}$ is stiff.

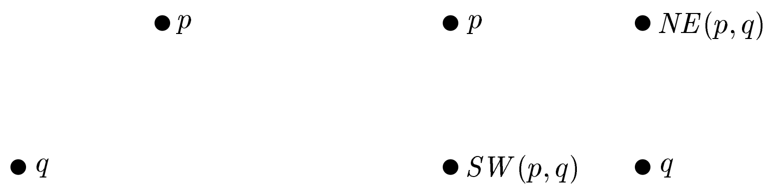
Finally, we have to show that $d_{\mathbf{P}}$ is extensible. To see this let $\mathcal{P}$ be a shortest path from p to q in $\mathbf{P}$. W.l.o.g. assume that p and q do not belong to $\partial\mathbf{P}$. Then, we can extend the first line segment s_1 and last line segment s_2 of $\mathcal{P}$ until they intersect the boundary of $\mathbf{P}$ in two points, say p_1 and q_1 (see Figure 4). We denote the extension of s_i by s'_i, for $i = 1,2$. Assume that $s'_1 = \overline{p_1p_2}$ with p_2 on $\mathcal{P}(p,q)$. Then, p belongs to $\mathcal{P}(p_1,p_2)$ and p_2 belongs to $\mathcal{P}(p,q)$. Hence, by Lemma 6.2 $\mathcal{P}(p_1,q)$ contains $\mathcal{P}(p,q)$. With a similar argument we obtain that $\mathcal{P}(p_1,q_1)$ contains $\mathcal{P}(p_1,q)$, and, thus, the first condition of extensibility is satisfied. The second condition follows trivially since we consider the inner metric of $\mathbf{P}$ and, hence, we can connect any two points with a $d_{\mathbf{P}}$-straight path in $\mathbf{P}$.

6.4 *NESW*-Convexity

NESW-convexity is a well studied semi-convex space that has been of special interest since it can be shown to satisfy the decomposition theorem of [RWar]. But, it fails to satisfy the join-condition.

It can be defined as follows. We say two points $p = (p_1,p_2)$ and $q = (q_1,q_2)$ in the plane are *comparable* if $(p_1 - q_1)(p_2 - q_2) \geq 0$; otherwise they are called *incomparable*. If p and q are incomparable and $p_1 < q_1$, we define the *NE-vertex* of p and q to be the point (q_1,p_2) and denote it by $NE(p,q)$ (see Figure 6). The *SW*-vertex $SW(p,q)$ of two incomparable points p and q is defined similarly as the point (p_1,q_2).

Definition 6.1 *Let* $\mathbf{C} \subseteq \mathbb{E}^2$ *and* p *and* q *be two points in* $\mathbf{C}$.



two comparable points the $NESW^*$-hull of two points

Figure 6: *The definition of NESW-convexity.*

(i) We say p and q *NE-see* (*SW-see*) each other in $\mathbf{C}$ if they are comparable or, if they are not, $NE(p, q)$ ($SW(p, q)$) is in $\mathbf{C}$.

(ii) $\mathbf{C}$ is *NE^*-convex* (*SW^*-convex*) if all points p and q in $\mathbf{C}$ *NE*-see (*SW-see*) each other.

We say that a set $\mathbf{C}$ *is* $NESW^*$-convex *if it is both NE^*- and SW^*-convex.*

Again the family of the $NESW^*$-convex sets forms a simple, complete semi-convex space. We first show that the $NESW^*$-convex sets do not satisfy the join-condition. In order to do so we consider the $NESW^*$-hull of two points. We have for two points p and q that

$$NESW^*\text{-}hull(\{p, q\}) = \begin{cases} \{p, q\} & \text{if } p \text{ and } q \\ & \text{comparable} \\ \{p, q, NE(p, q), SW(p, q)\} & \text{otherwise.} \end{cases}$$

To see that the join condition is not satisfied consider two points p and q such that p is to the left and below q. Therefore $\{p, q\}$ is $NESW^*$-convex and $NESW^*\text{-}hull(\{p, q\}) = \{p, q\}$. Let r be a point below p and to the right of q; then, $NESW^*\text{-}join(r, NESW^*\text{-}hull(\{p, q\}))$ consists of the union of $NESW^*\text{-}hull(\{r, p\})$ and $NESW^*\text{-}hull(\{r, q\})$ as shown in Figure 7. But it can be seen that $NE(p, SW(r, q))$ does not belong to $NESW^*\text{-}hull(\{r, p\}) \cup NESW^*\text{-}hull(\{r, q\})$ and, hence, $NESW^*\text{-}join(r, NESW^*\text{-}hull(\{p, q\}))$ is not $NESW^*$-convex.

We now turn to *NESW*-convexity as it was introduced in [SW84].

Definition 6.2 *Let* $\mathbf{C}$ *be a subset of the plane that consists of the connected components* $\mathbf{C}_i$*,* $i \in I$*.* $\mathbf{C}$ *is called NESW*-convex *if all* $\mathbf{C}_i$ *are $NESW^*$-convex, with* $i \in I$*.*

It can be easily seen that the family $\mathcal{C}$ of the *NESW*-convex sets is closed under intersection since the intersection of $NESW^*$-convex sets is *NESW*-convex. But $(\mathbb{E}^2, \mathcal{C})$ is not even aligned as the following example shows. Consider the family $\mathcal{N}$ of sets $\mathbf{C}_n$, $n \in \mathbb{N}$, which consist of the two rectangles $\mathbf{R}_n$ and $\mathbf{R}$ where $\mathbf{R}_n$ is spanned by the points $(0, 1)$ and $(1 - 1/n, 0)$ and $\mathbf{R}$ is spanned by the points $(1, 1)$ and $(2, -1)$ as shown in Figure 8a.

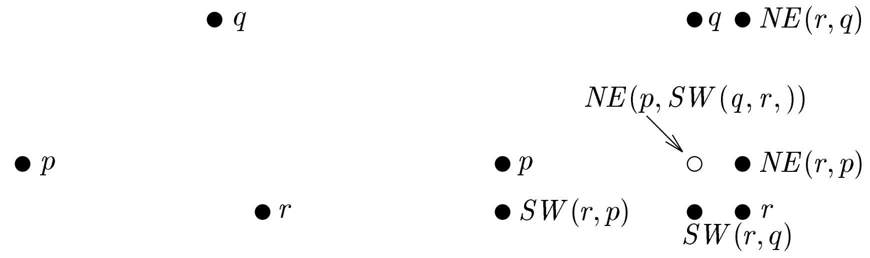


Figure 7: *NESW*-join$(p, $NESW*-hull$(\{q, r\}))$ is not NESW*-convex.*

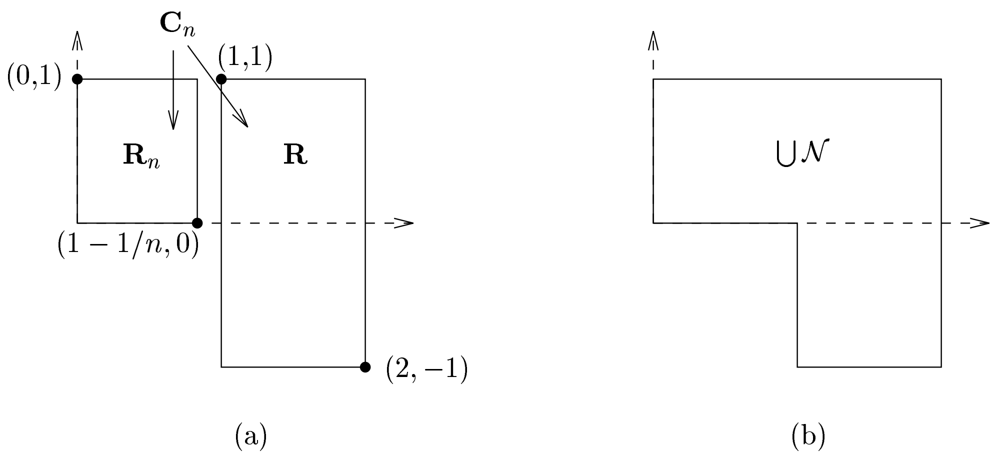


Figure 8: *A nested chain of NESW-convex sets.*

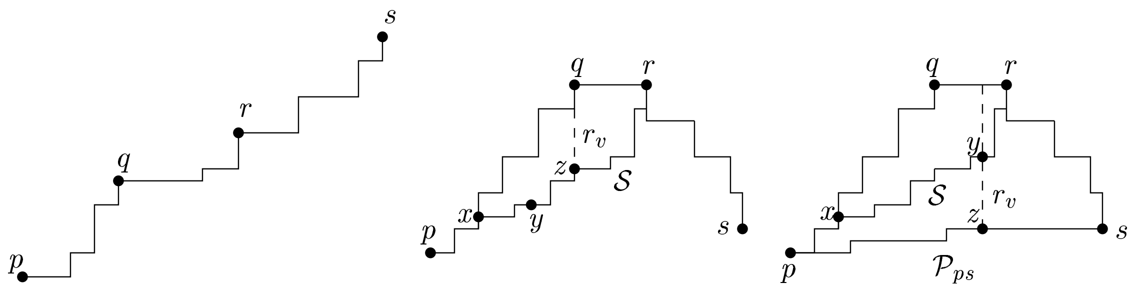


Figure 9: *Proving the stiffness of L_1.*

Clearly, $\mathcal{N}$ is a nested chain of *NESW*-convex sets; but the union of all sets in $\mathcal{N}$ is the connected L-shaped object in Figure 8b which is *not* $NESW^*$-convex. Hence, we can conclude by Lemma 3.10 that $(\mathbb{E}^2, \mathcal{C})$ does not satisfy the Kernel Theorem.

6.5 $\{0°, 90°\}^s$-convexity

Probably the most interesting example for us are the semi-convex spaces generated by the notion of $\{0°, 90°\}^s$-convexity. To define $\{0°, 90°\}^s$-convexity consider $\{0°, 90°\}$-convexity which we introduced above. A $\{0°, 90°\}$-convex path is called a $\{0°, 90°\}$-*stairsegment*. The connected components of $\{0°, 90°\}$-convex sets have the interesting property that each point can be connected to any other point by an $\{0°, 90°\}$-stairsegment that is completely contained in the component ([Raw87, Theorem 5.5.2]). We call a set that satisfies this last condition $\{0°, 90°\}^s$-*convex*. In order to make use of the results of Section 5 we note that the L_1-straight paths in the plane are just the $\{0°, 90°\}$-stairsegments. It is our aim to show that the $\{0°, 90°\}^s$-kernel of a simple polygon $\mathbf{P}$ in the plane, i.e., the points in $\mathbf{P}$ that can see every other point in $\mathbf{P}$ via a $\{0°, 90°\}$-stairsegment, is $\{0°, 90°\}^s$-convex. Since a simple polygon is a simple domain, we only have to show that L_1 satisfies being topological, extensible, and stiff. Once we have done this we conclude from the Kernel Theorem and Theorem 5.3 that the $\{0°, 90°\}^s$-kernel of any subset of $\mathbf{P}$ is $\{0°, 90°\}^s$-convex. This, in particular, implies that the $\{0°, 90°\}^s$-kernel of $\mathbf{P}$ is $\{0°, 90°\}^s$-convex.

So it only remains to show that L_1 satisfies the three properties of Theorem 5.3. That L_1 is topological follows immediately from the fact that L_1 is induced by a norm. We now consider the stiffness of L_1. Let p, q, r, and s be four points in the plane such that there is a $\{0°, 90°\}$-stairsegment $\mathcal{P}_{pr}$ from p to r that contains q, a $\{0°, 90°\}$-stairsegment $\mathcal{P}_{qs}$ from q to s that contains r and a $\{0°, 90°\}$-stairsegment $\mathcal{P}_{ps}$ from p to s (see Figure 9). W.l.o.g. assume that r is to the right and above p. If q is not on the same vertical or horizontal line segment as r, then r is also to the right and above q and, hence, any $\{0°, 90°\}$-stairsegment from q to r is from bottom left to top right. Therefore, s cannot be below or to the left of r and any $\{0°, 90°\}$-stairsegment from p to r gives rise to a $\{0°, 90°\}$-stairsegment from p to s by just adding the part of the $\{0°, 90°\}$-stairsegment from q to s which is after r.

Now assume that q is on the same horizontal line segment as r and that s is below and to the right of q. Consider a $\{0°, 90°\}$-stairsegment $\mathcal{S}$ from p to r. We have to show that $\mathcal{S}$ is contained in $\mathcal{C}_{L_1}$-$join(q, \mathcal{C}_{L_1}$-$hull(\{p, s\}))$. Let x be the last intersection point of $\mathcal{S}$ with $\mathcal{P}_{sp} * \mathcal{P}_{pr}$ on $\mathcal{S}$. Clearly, the part of $\mathcal{S}$ from p to x belongs to $\mathcal{C}_{L_1}$-$hull(\{p, q\}) \cup \mathcal{C}_{L_1}$-$hull(\{p, s\})$. Let y be some point on $\mathcal{S}$ after x. There are two cases to be considered. If y is to the left of q, then we consider the vertical ray r_v downward from q. Since r is above s and to the right of q and there are no intersection points of $\mathcal{S}$ with $\mathcal{P}_{ps}$ after q, the first path r_v intersects is $\mathcal{S}$. Let z be the intersection point. The boundary of $\mathbf{P}$ cannot cross the line segment l from q to z, since otherwise the intersection point would be enclosed by the curve $\mathcal{P}_{pr} * \mathcal{P}_{rs} * \mathcal{P}_{sq}$ which contradicts the fact that $\mathcal{P}_{pr} * \mathcal{P}_{rs} * \mathcal{P}_{sq}$ does not cross $\partial\mathbf{P}$ and $\mathbf{P}$ is simply connected. Since the part $\mathcal{S}_1$ of $\mathcal{S}$ from p to z stays below and to the left of z, $\mathcal{S}_1 * l$ is a $\{0°, 90°\}$-stairsegment from p to q. Hence, $y \in \mathcal{S}_1 * l \subseteq \mathcal{C}_{L_1}$-$hull(\{p, q\})$.

Now suppose that y is to the right of q. Since y is on a $\{0°, 90°\}$-stairsegment from p to r, y is to the left of r. Hence, there is a point z on the horizontal line segment from q to r that has the same x-coordinate as y. We denote the line segment from q to z by l_1. Consider the vertical ray r_v downward starting in z. By the same argument as above we have that r_v intersects $\mathcal{S}$ in y before it intersects $\mathcal{P}_{ps}$ and that $\partial\mathbf{P}$ does not cross the line segment l_2 from z to $\mathcal{P}_{ps} \cap r_v$. Hence, y belongs to the path consisting of $l_1 * l_2$ which is part of $\mathcal{C}_{L_1}$-$join(q, \mathcal{C}_{L_1}$-$hull(\{p, s\}))$. This completes the proof.

We now show the extensibility of L_1. Let $\mathcal{S}$ be a d-straight path from p to q in $\mathbf{P}$. W.l.o.g. assume that q is to the right and above p. We extend $\mathcal{S}$ by two horizontal line segments, one to the left of p and one to the right of q that intersect the boundary of $\mathbf{P}$. It is easily seen that $\mathcal{S}$ stays $\{0°, 90°\}$-convex if we append the line segments.

The last property we have to show is that all the points in a L_1-triangle $\mathbf{T}$ can be connected to the corner points with a L_1-straight path that is completely contained in $\mathbf{T}$. In order to do so, we make use of the following powerful result by R. Menger ([Rin61, p. 148,17.5]).

Theorem 6.3 (Verbindbarkeitssatz) *If $\mathbf{X}$ is metrically complete and, for all points $p, q \in \mathbf{X}$, there exists a third point $r \in \mathbf{X}$ with $d(p, r) + d(r, q) = d(p, q)$, then any two points can be connected by a d-straight path contained in $\mathbf{X}$.*

Hence, consider a L_1-triangle $\mathbf{T}$ which is bounded by the curves $\mathcal{P}_{pq}$, $\mathcal{P}_{qr}$, and $\mathcal{P}_{rp}$ and two points x and y in $\mathbf{T}$. We want to show that there is at least one point z in $\mathbf{T}$ that lies between x and y, i.e., z satisfies the equation $L_1(x, z) + L_1(z, y) = L_1(x, y)$. First, if x is an inner point of $\mathbf{T}$, then since the set of all points that lie between x and y is just the rectangle $\mathbf{R}$ spanned by x and y and there is some neighbourhood N of x that is contained in $\mathbf{T}$, $N \cap \mathbf{R}$ contains at least one point different from x that lies between x and y. Hence, we can assume that x and y are on $\mathcal{P}_{pq} * \mathcal{P}_{qr} * \mathcal{P}_{rp}$. Since the paths are L_1-straight, we can assume that x and y are on two different paths, say $\mathcal{P}_{pq}$ and $\mathcal{P}_{pr}$. W.l.o.g. assume that y is above and to the right of x and that x is no corner point. Since y is on the same side of $\mathcal{P}_{pq}$ as $int(\mathbf{T})$, there is some part of the horizontal ray to the right starting in x that belongs to $\mathbf{T}$.

Hence, for every two points x and y in $\mathbf{T}$, there exists a point that lies between them. By Theorem 6.3 we then have that any two points in $\mathbf{T}$ can be connected by a L_1-straight path.

Appendix: Proving Theorem 5.3

In Section 5 we stated the following theorem, which we now prove.

Theorem 5.3 *If* $\mathbf{X}$ *is a simple domain and* d *is an extensible, stiff, and topological metric in* $\mathbf{X}$*, then* $(\mathbf{X}, \mathcal{C}_d)$ *satisfies the join-property.*

Since we already have illustrated the main proof idea, we start with the formal proof right away. So, from now on, $\mathbf{X}$ will denote a simple domain and d an extensible, stiff, and topological metric (except if stated otherwise).

6.6 Definitions

We will be mostly dealing with d-straight paths in the plane. So we need to introduce some notation concerning paths and subpaths.

We consider a *curve* or *path* to be an equivalence class $\mathcal{C}$ of continuous mappings $\varphi : I \longrightarrow I\!\!E^2$ where I is a real interval. $\varphi \in \mathcal{C}$ is called a *parametrization of* $\mathcal{C}$. We will mostly view a curve as a point set but it should be kept in mind that a curve can always be endowed with a parametrization. Given two curves $\mathcal{P}$ and $\mathcal{Q}$ with two parametrizations $\varphi_{\mathcal{P}} : \langle a, b] \longrightarrow I\!\!E^2$ and $\varphi_{\mathcal{Q}} : [c, d\rangle \longrightarrow I\!\!E^2$ (the brackets $\langle$ and $\rangle$ are used to indicate that it doesn't matter if the left or right end point belongs to the interval or not) such that $\varphi_{\mathcal{P}}(b) = \varphi_{\mathcal{Q}}(c)$ we define the *concatenation of* $\mathcal{P}$ *and* $\mathcal{Q}$, denoted by $\mathcal{P} * \mathcal{Q}$, as the equivalence class of the mapping $\varphi_{\mathcal{P}*\mathcal{Q}} : \langle a, d - c + b\rangle \longrightarrow I\!\!E^2$ which is given by

$$\varphi_{\mathcal{P}*\mathcal{Q}}(x) = \begin{cases} \varphi_{\mathcal{P}}(x) & \text{if } x \in \langle a, b] \\ \varphi_{\mathcal{Q}}(x + c - b) & \text{if } x \in [b, d - c + b\rangle. \end{cases}$$

We call a simple closed curve according to [Rin75] a *Jordan curve*. By the famous Jordan Curve Theorem [Rin75, 40.1] a simple closed curve $\mathcal{J}$ partitions the plane into two parts, one bounded called *the interior of* $\mathcal{J}$ and denoted by $int(\mathcal{J})$, and the other unbounded called *the exterior of* $\mathcal{J}$ and denoted by $ext(\mathcal{J})$. We define the *domain* of $\mathcal{J}$ to be the union of $int(\mathcal{J})$ and $\mathcal{J}$ and denote it by $dom(\mathcal{J})$.

Consider two points p and q in $\mathbf{X}$. In the following we will assume that $\mathcal{C}_d$-$hull(\{p, q\})$ is non-empty. By $\mathcal{P}_{pq}$ we denote a d-straight path from p to q. We assume that $\mathcal{P}_{pq}$ is endowed with an injective parametrization that has p as start point and q as end point. So, if we talk of points on $\mathcal{P}_{pq}$ being before or after others, this is meant according to the order induced by the parametrization. If r and s are two points on $\mathcal{P}_{pq}$, we denote the part of $\mathcal{P}_{pq}$ from r to s by $\mathcal{P}_{pq}|_{rs}$. It is important to note that the orientation of $\mathcal{P}_{pq}|_{rs}$ depends on the relative position of r and s on $\mathcal{P}_{pq}$ and not on p and q. So if $c : [0, 1] \longrightarrow \mathbf{X}$ is a parametrization of $\mathcal{P}_{pq}$ with $c(t_1) = r$ and $c(t_2) = s$ and $t_2 < t_1$, then $\mathcal{P}_{pq}|_{rs}$ is supposed to have the parametrization $c' : [1 - t_1, 1 - t_2] \longrightarrow \mathbf{X}$ with $c'(t) = c(1 - t)$.

Since d is extensible, we can extend a d-straight path $\mathcal{P}$ to a d-straight path $\mathcal{P}'$ that contains $\mathcal{P}$ and splits $\mathbf{X}$ into at least two components. $\mathcal{P}'$ is called an *extension* of $\mathcal{P}$. We denote an extension of a d-straight path $\mathcal{P}$ by $\overline{\mathcal{P}}$.

Recall that d-straight paths are shortest paths in $\mathbf{X}$ [Rin61, p. 147]. If there exists a d-straight path from p to q, then any other shortest path from p to q is also d-straight. If r and s are two points on $\mathcal{P}_{pq}$, then any shortest path $\mathcal{P}_{rs}$ from r to s is also d-straight and so is $\mathcal{P}_{pq}|_{pr} * \mathcal{P}_{rs} * \mathcal{P}_{pq}|_{sq}$ if r occurs before s on $\mathcal{P}_{pq}$ [Rin61, p. 147]. Because of the positive definiteness of metrics d-straight paths are always simple. Since we need to show that a path is the shortest path from its start point to its end point, we have to refer to the length of a path. So, if $\mathcal{P}$ is a path, then $\lambda_d(\mathcal{P})$ denotes its length. Note that if two d-straight paths $\mathcal{P}$ and $\mathcal{Q}$ start and end in the same point, then we have $\lambda_d(\mathcal{P}) = \lambda_d(\mathcal{Q})$. These are the main definitions we will use in proving Theorem 5.3.

We also need the following property of curves in the plane (see [Kle89, Lemma 2.3.1]).

Lemma 6.4 *Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be simple curves in $\mathbb{E}^2$ leading from a common point x to points $p_1, \ldots, p_m$, respectively, such that $\mathcal{C}_i \cap \mathcal{C}_j = x$ holds if $i \neq j$. Then, for each neighbourhood N of x, there exists an open neighbourhood V of x such that*

(i) $\overline{V} \subseteq N$;

(ii) ∂V is a Jordan curve;

(iii) $\partial V \cap \mathcal{C}_i$ consists of exactly one point, for all $1 \leq i \leq m$.

Proof: See [Rin75, 40.20]. □

A last observation we need before going into the proof of Theorem 5.3 concerns the intersection points of d-straight paths. We will often have to deal with choosing a first or last intersection point of two curves. In order to do so consistently, we have to ensure that the ordering of the intersection points is the same on both considered curves. This is, of course, not true in general, but if the curves start in a common point, then the following lemma shows that the notion of first and last is independent of the curve we consider.[9]

Lemma 6.5 *Let $\mathcal{P}_{xy}$ and $\mathcal{P}_{xz}$ be two d-straight paths. If s is the first resp. last intersection point of $\mathcal{P}_{xy}$ and $\mathcal{P}_{xz}$ on $\mathcal{P}_{xy}$, then s is also the first resp. last intersection point of $\mathcal{P}_{xy}$ and $\mathcal{P}_{xz}$ on $\mathcal{P}_{xz}$.*

Proof: Let s be the first intersection point of $\mathcal{P}_{xy}$ and $\mathcal{P}_{xz}$ on $\mathcal{P}_{xy}$. Assume there is another intersection point r of $\mathcal{P}_{xy}$ and $\mathcal{P}_{xz}$ before s on $\mathcal{P}_{xz}$. Clearly, r is after s on $\mathcal{P}_{xy}$. Hence,

$$\begin{aligned}
\lambda_d(\mathcal{P}_{xy}|_{xs}) &= \lambda_d(\mathcal{P}_{xz}|_{xs}) \\
&= \lambda_d(\mathcal{P}_{xz}|_{xr}) + \lambda_d(\mathcal{P}_{xz}|_{rs}) \\
&= \lambda_d(\mathcal{P}_{xy}|_{xr}) + \lambda_d(\mathcal{P}_{xz}|_{rs}) \\
&= \lambda_d(\mathcal{P}_{xy}|_{xs}) + \lambda_d(\mathcal{P}_{xy}|_{sr}) + \lambda_d(\mathcal{P}_{xz}|_{rs})
\end{aligned}$$

[9] Note that because of the compactness of the curves we consider we are allowed to use the terms "first" or "last" without addressing the question of their existence.

which, of course, implies that $s = r$ and, therefore, contradicts our assumption.

A similar computation yields the result for the last intersection point. $\square$

The above lemma justifies that from now on we will dispense with specifying on which path an intersection is first or last if the concerned paths start at the same point. Note, that the meaning of first and last is, of course, reverted if both paths end in the same point.

6.7 Some Properties of the $\mathcal{C}_d$-*hull*

The $\mathcal{C}_d$-*join* depends on the properties of the $\mathcal{C}_d$-*hull*. Hence, we will first show that the $\mathcal{C}_d$-*hull* satisfies the definition of a convex hull.

Lemma 6.6 *Let d be a topological and extensible metric and x, y be two points in $\mathbf{X}$ with non-empty $\mathcal{C}_d$-hull. If $p, q \in \mathcal{C}_d$-hull$(\{x, y\})$, then $\mathcal{C}_d$-hull$(\{p, q\})$ is non-empty and $\mathcal{C}_d$-hull$(\{p, q\}) \subseteq \mathcal{C}_d$-hull$(\{x, y\})$.*

Proof: Let $\mathcal{C}_p$, $\mathcal{C}_q$ be two d-straight curves from x to y with $p \in \mathcal{C}_p$ and $q \in \mathcal{C}_q$. Let $\overline{\mathcal{C}}_p$ be an extension of $\mathcal{C}_p$ that splits $\mathbf{X}$ into two or more components. We have to show that there is at least one d-straight path in $\mathbf{X}$ from p to q and that all d-straight paths from p to q that are contained in $\mathbf{X}$ belong to $\mathcal{C}_d$-*hull*$(\{x, y\})$.

First assume that $q \in \overline{\mathcal{C}}_p$. If $q \in \mathcal{C}_p$, then $\mathcal{C}_d$-*hull*$(\{p, q\}) \subseteq \mathcal{C}_d$-*hull*$(\{x, y\})$ since any d-straight path $\mathcal{P}_{pq}$ from p to q gives rise to two paths $\mathcal{C}_p|_{xp} * \mathcal{P}_{pq} * \mathcal{C}_p|_{qy}$ or $\mathcal{C}_p|_{xq} * \mathcal{P}_{pq}|_{qp} * \mathcal{C}_p|_{py}$ one of which is a d-straight path from x to y depending on which point occurs first on $\mathcal{C}_p$. Furthermore, $\mathcal{C}_p|_{pq}$ is a d-straight path from p to q in $\mathbf{X}$.

On the other hand, if $q \notin \mathcal{C}_p$, we can assume w.l.o.g. that q occurs before x on $\overline{\mathcal{C}}_p$ and, thus,

$$
\begin{aligned}
\lambda_d(\mathcal{C}_q|_{yq}) &= \lambda_d(\overline{\mathcal{C}}_p|_{yq}) \\
&= \lambda_d(\mathcal{C}_p|_{yx}) + \lambda_d(\overline{\mathcal{C}}_p|_{xq}) \\
&> \lambda_d(\mathcal{C}_p|_{yx}) \\
&= \lambda_d(\mathcal{C}_q|_{yx}) \\
&> \lambda_d(\mathcal{C}_q|_{yq}).
\end{aligned}
$$

The first equality holds since both $\mathcal{C}_q|_{yq}$ and $\overline{\mathcal{C}}_p|_{yq}$ are d-straight and thus shortest paths from y to q. The above inequality is, of course, a contradiction; thus, we cannot have $q \in \overline{\mathcal{C}}_p \setminus \mathcal{C}_p$.

Therefore, assume that $q \notin \overline{\mathcal{C}}_p$. Hence, there is a component, say $\mathbf{Q}_1$, that contains q. Let $\mathcal{P}_{pq}$ be a d-straight path in $\mathbb{E}^2$ that connects p and q. We have to show that if $\mathcal{P}_{pq} \subseteq \mathbf{X}$, then there is a d-straight path from x to y through every point of $\mathcal{P}_{pq}$. Furthermore, we have to show that there is at least one d-straight path $\mathcal{P}'_{pq}$ from p to q that is contained in $\mathbf{X}$.

Let p_1 be the last intersection of $\mathcal{P}_{pq}$ with $\mathcal{C}_p$ on $\mathcal{P}_{pq}$ and q_1 the first intersection point of $\mathcal{P}_{pq}$ with $\mathcal{C}_q$ on $\mathcal{P}_{pq}$. $\mathcal{C}_p|_{xp} * \mathcal{P}_{pq}|_{pp_1} * \mathcal{C}_p|_{p_1y}$ is a d-straight path from x to y and, hence, if $\mathcal{P}_{pq}$ is contained in $\mathbf{X}$, then $\mathcal{P}_{pq}|_{pp_1} \subseteq \mathcal{C}_d\text{-}hull(\{x,y\})$. For the same reason we have $\mathcal{P}_{pq}|_{q_1q} \subseteq \mathcal{C}_d\text{-}hull(\{x,y\})$.

On the other hand, $\mathcal{C}_p|_{pp_1}$ and $\mathcal{C}_q|_{qq_1}$ are d-straight paths in $\mathbf{X}$ and, hence, we only have to show that there is at least one d-straight path from p_1 to q_1 in $\mathbf{X}$.

We claim that p_1 occurs before q_1 on $\mathcal{P}_{pq}$. To see this assume the contrary. Then, we have that

$$\lambda_d(\mathcal{P}_{pq}|_{pp_1}) = \lambda_d(\mathcal{P}_{pq}|_{pq_1}) + \lambda_d(\mathcal{P}_{pq}|_{q_1p_1})$$

and

$$\begin{aligned}\lambda_d(\mathcal{C}_p) &= \lambda_d(\mathcal{C}_p|_{xp}) + \lambda_d(\mathcal{C}_p|_{pp_1}) + \lambda_d(\mathcal{C}_p|_{p_1y}) \\ &= \lambda_d(\mathcal{C}_p|_{xp}) + \lambda_d(\mathcal{P}_{pq}|_{pp_1}) + \lambda_d(\mathcal{C}_p|_{p_1y}) \\ &= \lambda_d(\mathcal{C}_p|_{xp}) + \lambda_d(\mathcal{P}_{pq}|_{pq_1}) + \lambda_d(\mathcal{P}_{pq}|_{q_1p_1}) + \lambda_d(\mathcal{C}_p|_{p_1y}).\end{aligned}$$

But we also have that

$$\lambda_d(\mathcal{C}_p) = \lambda_d(\mathcal{C}_p|_{xp}) + \lambda_d(\mathcal{P}_{pq}|_{pq_1}) + \lambda_d(\mathcal{C}_q|_{q_1y}).$$

Hence,

$$\lambda_d(\mathcal{C}_q|_{q_1y}) = \lambda_d(\mathcal{P}_{pq}|_{q_1p_1}) + \lambda_d(\mathcal{P}_{pq}|_{p_1y})$$

With a similar computation we obtain that

$$\lambda_d(\mathcal{C}_p|_{p_1y}) = \lambda_d(\mathcal{P}_{pq}|_{p_1q_1}) + \lambda_d(\mathcal{P}_{pq}|_{q_1y})$$

and, therefore, $\lambda_d(\mathcal{P}_{pq}|_{p_1q_1}) = 0$, i.e., $p_1 = q_1$ and we are done.

Therefore, we can assume that $\mathcal{P} = \mathcal{P}_{pq}|_{p_1q_1}$ is non-empty and does neither intersect $\mathcal{C}_p$ nor $\mathcal{C}_q$. Now let x_1 be the last intersection point of $\mathcal{C}_p$ and $\mathcal{C}_q$ before p_1 and q_1 and y_1 the first intersection point of $\mathcal{C}_p$ and $\mathcal{C}_q$ after p_1 and q_1. We distinguish two cases (see Figure 10).

Case 1 There is an intersection point x_2 of $\mathcal{C}_p$ and $\mathcal{C}_q$ between x_1 and y_1.
By the definition of x_1 and y_1, x_2 is between p_1 and q_1. W.l.o.g. let p_1 be after x_2 and q_1 be before it. Since $\lambda_d(\mathcal{C}_p|_{xx_2}) = \lambda_d(\mathcal{C}_q|_{xx_2})$ and $\lambda_d(\mathcal{C}_p|_{x_2y}) = \lambda_d(\mathcal{C}_q|_{x_2y})$, $\mathcal{C}_q|_{xx_2} * \mathcal{C}_p|_{x_2y}$ is a d-straight path from x to y that contains p_1 and q_1. Hence, $\mathcal{C}_d\text{-}hull(\{p_1,q_1\})$ is non-empty. Furthermore, $\mathcal{C}_q|_{xq_1} * \mathcal{P} * \mathcal{C}_p|_{p_1y}$ is also d-straight and $\mathcal{P} \subseteq \mathcal{C}_d\text{-}hull(\{x,y\})$.

Case 2 $\mathcal{C} = \mathcal{C}_p|_{x_1y_1} * \mathcal{C}_q|_{y_1x_1}$ is a Jordan curve.
We claim that $\mathcal{P}$ is contained in the interior of $\mathcal{C}$. This immediately implies that $\mathcal{P}$ is contained in $\mathbf{X}$ and that $\mathcal{C}_d\text{-}hull(\{p,q\})$ is non-empty.

To see this we Assume that it is not the case. Since $int(\mathcal{C})$ is path-connected and $int(\mathcal{C})$ does not intersect $\overline{\mathcal{C}}_p$, $int(\mathcal{C})$ belongs entirely to one of the components $\mathbf{X}$ is split into by

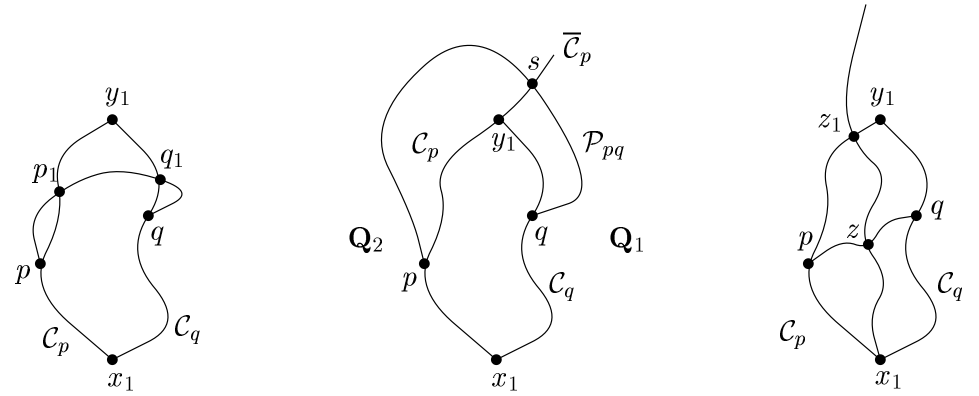


Figure 10: *Illustration of Case 2.*

$\overline{\mathcal{C}}_p$. We can connect every point in $int(\mathcal{C})$ to q by a path without crossing $\overline{\mathcal{C}}_p$; therefore, $int(\mathcal{C})$ belongs also to $\mathbf{Q}_1$. Since $p_1 \in \mathcal{C} \cap \overline{\mathcal{C}}_p \setminus \{x, y\}$, there is a neighbourhood N of p_1 such that $N \cap \overline{\mathcal{C}}_p = N \cap \mathcal{C}$. By Lemma 6.4 we can choose N such that ∂N is a Jordan curve and $\overline{\mathcal{C}}_p$ intersects ∂N only twice. $\overline{\mathcal{C}}_p$ splits N into two components. Since $int(\mathcal{C}) \subseteq \mathbf{Q}_1$ and $\partial \mathbf{Q}_1 \cap N = \partial int(\mathcal{C}) \cap N$, we have $\mathbf{Q}_1 \cap N = int(\mathcal{C}) \cap N$. This implies that $ext(\mathcal{C}) \cap N = \mathbf{Q}_2 \cap N$ where $\mathbf{Q}_2$ is a component different from $\mathbf{Q}_1$. Since all the points of $\mathcal{P}$ are contained in $ext(\mathcal{C})$, $\mathcal{P}$ also intersects $ext(\mathcal{C})$ in N and, hence, $\mathbf{Q}_2$. Since $\mathcal{P}$ also intersects $\mathbf{Q}_1$, we have that $\mathcal{P}$ crosses $\overline{\mathcal{C}}_p$ either before x_1 or after y_1 in a point s. W.l.o.g. assume that s is after y_1 on $\overline{\mathcal{C}}_p$. Then, $\lambda_d(\overline{\mathcal{C}}_p|_{x_1 s}) = \lambda_d(\mathcal{C}_p|_{x_1 y_1}) + \lambda_d(\overline{\mathcal{C}}_p|_{y_1 s})$ and this equals $\lambda_d(\mathcal{C}_q|_{x_1 q_1}) + \lambda_d(\mathcal{C}_q|_{q_1 y_1}) + \lambda_d(\overline{\mathcal{C}}_p|_{y_1 s})$. Hence, $\mathcal{C}_q|_{q_1 y_1} * \overline{\mathcal{C}}_p|_{y_1 s}$ is d-straight and $\lambda_d(\mathcal{P}_{pq}|_{q_1 s}) = \lambda_d(\overline{\mathcal{C}}_q|_{q_1 y_1}) + \lambda_d(\overline{\mathcal{C}}_p|_{y_1 s})$. Since we also have $\lambda_d(\mathcal{P}_{pq}|_{p_1 s}) = \lambda_d(\mathcal{C}_p|_{p_1 y_1}) + \lambda_d(\overline{\mathcal{C}}_p|_{y_1 s})$ and $\mathcal{C}_p|_{p_1 y_1} * \mathcal{C}_q|_{y_1 q_1}$ is a path from p_1 to q_1 which is $2\lambda_d(\overline{\mathcal{C}}_p|_{y_1 s})$ shorter than $\mathcal{P}$, $d(y_1, s)$ equals 0. This, of course, immediately implies that $y_1 = s$ and $\mathcal{P}_{pq}|_{p_1 y_1} \subseteq \mathcal{C}_d\text{-}hull(\{x, y\})$ as well as $\mathcal{P}_{pq}|_{y_1 q_1} \subseteq \mathcal{C}_d\text{-}hull(\{x, y\})$.

Hence, we can assume that if $z \in \mathcal{P}$, then z is contained in the interior of $\mathcal{C}$. Since $\mathcal{C}$ is d-triangle, we can connect x_1 by a d-straight path $\mathcal{P}_1$ to z. Now $\mathcal{C}$ is a Jordan curve and z is contained in the interior of $\mathcal{C}$, therefore, an extension $\overline{\mathcal{P}}_1$ of $\mathcal{P}_1$ intersects $\mathcal{C}$ in some point z_1.

If z_1 belongs to $\mathcal{C}_p$, then $\mathcal{C}_p|_{x x_1} * \overline{\mathcal{P}}_1|_{x_1 z_1} * \mathcal{C}_p|_{z_1 y}$ is a d-straight path from x to y and $\mathcal{P}_1 \subseteq \mathcal{C}_d\text{-}hull(\{x, y\})$. A similar statement holds if z_1 belongs to $\mathcal{C}_q$.

□

The above proof gives a good impression of the techniques we will make use of in proving the main theorem.

6.8 *d*-Triangles and Jordan Curves

Before we start to take a look at the proof of the theorem we make some general observations about d-triangles. This section is crucial to the rest of the proof since the situation we

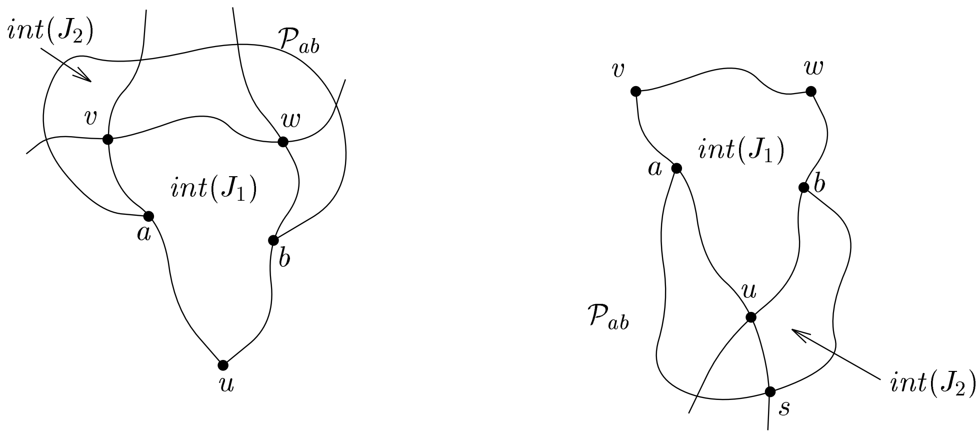


Figure 11: *The possible intersections of $\mathcal{P}_{ab}$.*

consider here will arise in a number of different cases.

Let u, v, and w be three points in $\mathbf{X}$ and $\mathcal{P}_{uv}$, $\mathcal{P}_{vw}$, and $\mathcal{P}_{uw}$ be three d-straight paths connecting them such that $\mathcal{P}_{uv} * \mathcal{P}_{vw} * \mathcal{P}_{uw}$ is a Jordan curve $\mathcal{J}_1$. Let $a \in \mathcal{P}_{uv} \setminus \{u\}$ and $b \in \mathcal{P}_{uw} \setminus \{u, w\}$. Further, let the curve $\mathcal{P}_{ab}$ from a to b be contained in the exterior of $\mathcal{J}_1$. Note that we do not require $\mathcal{P}_{ab}$ to be d-straight. Let $\mathcal{J}_2$ be the Jordan-curve $\mathcal{P}_{uv}|_{ua} * \mathcal{P}_{ab} * \mathcal{P}_{uw}|_{bu}$.

In the following it is our aim to show that $\mathcal{P}_{ab}$ intersects $\overline{\mathcal{P}}_{uv}$ or $\overline{\mathcal{P}}_{uw}$ or $\overline{\mathcal{P}}_{vw}$ since the intersection points constructed in this section are needed later to deal with degenerate cases (see Figure 11). We note that if v does not equal a, then either both v and w are in $int(\mathcal{J}_2)$ or in $ext(\mathcal{J}_2)$ since $\mathcal{P}_{vw}$ connects v to w without intersecting $\mathcal{J}_2$. Hence, v and w belong to the same connected component. This implies that there are two cases, we have to distinguish. Either w is in $int(\mathcal{J}_2)$ or in $ext(\mathcal{J}_2)$. First assume that w is in $int(\mathcal{J}_2)$.

Lemma 6.7 *Let w be in $int(\mathcal{J}_2)$. Either $\mathcal{P}_{ab}$ intersects $\overline{\mathcal{P}}_{uw}$ and $\overline{\mathcal{P}}_{vw}$ after w or $\mathcal{C}_d$-join$(u, \mathcal{C}_d\text{-}hull(\{v, w\}))$ is d-convex. If a does not equal v, then either $\mathcal{P}_{ab}$ intersects $\overline{\mathcal{P}}_{uv}$ and $\overline{\mathcal{P}}_{vw}$ after v or $\mathcal{C}_d$-join$(u, \mathcal{C}_d\text{-}hull(\{v, w\}))$ is d-convex.*

Proof: Since w is in $int(\mathcal{J}_2)$ but the part of $\overline{\mathcal{P}}_{uw}$ after w cannot be contained in a Jordan-curve, $\overline{\mathcal{P}}_{uw}$ intersects $\mathcal{J}_2$ in some point s. Assume that s is not in $\mathcal{P}_{ab}$. Since $\overline{\mathcal{P}}_{uw}$ is simple and $\mathcal{J}_2 \setminus \mathcal{P}_{ab} = \mathcal{P}_{uv}|_{au} * \mathcal{P}_{uw}|_{ub}$, $\overline{\mathcal{P}}_{uw}$ intersects $\mathcal{P}_{uv}$ in s. But then $w \in \mathcal{C}_d\text{-}hull(\{s, u\})$ and $s \in \mathcal{C}_d\text{-}hull(\{u, v\})$ and, hence, $w \in \mathcal{C}_d\text{-}hull(\{u, v\})$. This, of course, implies that $\mathcal{C}_d$-join$(u, \mathcal{C}_d\text{-}hull(\{v, w\}))$ is d-convex. The other claims follows similarly. $\square$

Now assume that w is in $ext(\mathcal{J}_1)$. Since we can connect $w \in ext(\mathcal{J}_2)$ to any point in $int(\mathcal{J}_1)$ by a curve that is completely contained in $int(\mathcal{J}_1)$ and, thus, does not cross $\mathcal{J}_2$, $int(\mathcal{J}_1)$ is also contained in $ext(\mathcal{J}_2)$. Choose a sufficiently small circular neighbourhood N of u such that $N \cap \mathcal{J}_1 = N \cap \mathcal{J}_2$ and ∂N meets $\mathcal{J}_1$ ($\mathcal{J}_2$) only in two points which is possible by Lemma 6.4. Since $int(\mathcal{J}_1) \subseteq ext(\mathcal{J}_2)$, we have $N \cap ext(\mathcal{J}_2) = N \cap int(\mathcal{J}_1)$. So any path $\mathcal{S}$ originating in u that cannot be contained in a Jordan-curve must intersect the exterior of $\mathcal{J}_2$ and $\mathcal{J}_1$. Since $\mathcal{P}_{uv}|_{au} * \mathcal{P}_{uw}|_{ub}$ belongs to both $\mathcal{J}_2$ and $\mathcal{J}_1$ and the interior of $\mathcal{J}_1$ is contained in the exterior of $\mathcal{J}_2$, $\mathcal{S}$ intersects the curve $\mathcal{P}_{ab} * \mathcal{P}_{uw}|_{bw} * \mathcal{P}_{vw} * \mathcal{P}_{uv}|_{va}$. This leads to the following lemma.

Lemma 6.8 *If w is in $ext(\mathcal{J}_2)$, then the part of $\overline{\mathcal{P}}_{uv}$ ($\overline{\mathcal{P}}_{uw}$) before u intersects $\mathcal{P}_{ab}$ or it intersects $\mathcal{P}_{uw}|_{bw}$ ($\mathcal{P}_{uv}|_{va}$) or $\mathcal{C}_d$-$join(u, \mathcal{C}_d\text{-}hull(\{v,w\}))$ is d-convex.*

Proof: Since the part of the extension of $\mathcal{P}_{uv}$ ($\mathcal{P}_{uw}$) before u cannot be contained in a Jordan-curve and intersects $int(\mathcal{J}_1)$ or $int(\mathcal{J}_2)$, it crosses one of $\mathcal{J}_1$ and $\mathcal{J}_2$. Assume the part of $\overline{\mathcal{P}}_{uv}$ before u does not intersect $\mathcal{P}_{ab}$. (The argument for $\overline{\mathcal{P}}_{uw}$ is similar). Then, $\overline{\mathcal{P}}_{uv}$ intersects $\mathcal{P}_{uv}|_{av} * \mathcal{P}_{vw} * \mathcal{P}_{uw}|_{wb}$ in some point s. Since $\overline{\mathcal{P}}_{uv}$ is simple, s cannot be in $\mathcal{P}_{uv}|_{av}$. Assume that $\overline{\mathcal{P}}_{uv}$ does not intersect $\mathcal{P}_{uw}|_{wb}$. Hence, s is in $\mathcal{P}_{vw}$. Since $u \in \mathcal{C}_d\text{-}hull(\{v,s\})$, we have $u \in \mathcal{C}_d\text{-}hull(\{v,w\})$ and $\mathcal{C}_d\text{-}join(u, \mathcal{C}_d\text{-}hull(\{v,w\})) = \mathcal{C}_d\text{-}hull(\{v,w\})$,i.e., $s \in \mathcal{C}_d\text{-}hull(\{v,w\})$. □

Finally we need the following lemma which is useful to show that certain compound paths are d-straight.

Lemma 6.9 *If the part of $\overline{\mathcal{P}}_{uv}$ before u intersects $\mathcal{P}_{ab}$ in s and $\mathcal{P}_{ab}$ is a d-straight path, then $\lambda_d(\mathcal{P}_{uw}|_{ub}) = \lambda_d(\overline{\mathcal{P}}_{uv}|_{us} * \mathcal{P}_{ab}|_{sb})$.*

Proof: For illustration see Figure 11. We have

$$\begin{array}{rcl} \lambda_d(\mathcal{P}_{ab}) & = & \lambda_d(\mathcal{P}_{ab}|_{as}) + \lambda_d(\mathcal{P}_{ab}|_{sb}) \\ & = & \lambda_d(\overline{\mathcal{P}}_{uv}|_{au}) + \lambda_d(\overline{\mathcal{P}}_{uv}|_{us}) + \lambda_d(\mathcal{P}_{ab}|_{sb}) \\ & \geq & \lambda_d(\overline{\mathcal{P}}_{uv}|_{au}) + \lambda_d(\mathcal{P}_{uw}|_{ub}) \\ & \geq & \lambda_d(\mathcal{P}_{ab}) \end{array}$$

and, hence, $\lambda_d(\overline{\mathcal{P}}_{uv}|_{us}) + \lambda_d(\mathcal{P}_{ab}|_{sb}) = \lambda_d(\mathcal{P}_{uw}|_{ub})$ □

6.9 Proving the Theorem

As we already have pointed out before, the proof of Theorem 5.3 consist mainly of considering special cases which have to be dealt with because of the very flexible nature of d-straight curves. The proof is organized as follows. In order to reduce the number of cases as far as possible we first make a few observations that restrict the combinatorial complexity of the situation we consider. We then turn to the four main cases of the proof.

6.9.1 Reducing the General Case

Since the join only depends on three points, let x, y, and z be three points in $\mathbf{X}$ with non-empty join, i.e., $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\})) \neq \emptyset$. Further, let $p, q \in \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ and $\mathcal{P}_{pq}$ be a d-straight path from p to q. We have to show that

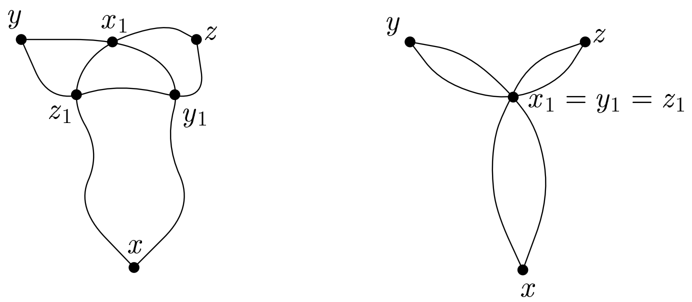


Figure 12: *x_1 is after y_1 on $\mathcal{P}_{xy}$ or z_1 is before y_1 on $\mathcal{P}_{yz}$.*

$\mathcal{P}_{pq} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$. Since p and q are in $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$, there are points y_p and z_q in $\mathcal{C}_d\text{-}hull(\{y,z\})$ with $p \in \mathcal{C}_d\text{-}hull(\{x, y_p\})$ and $q \in \mathcal{C}_d\text{-}hull(\{x, z_q\})$. By Lemma 6.6 we have $\mathcal{C}_d\text{-}hull(\{y_p, z_q\}) \subseteq \mathcal{C}_d\text{-}hull(\{y,z\})$. Hence, it suffices to show that $\mathcal{P}_{pq}$ is contained in $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y_p, z_q\}))$. So w.l.o.g. we assume from now on that $y = y_p$ and $z = z_q$.

Let $\mathcal{P}_{xy}$ and $\mathcal{P}_{xz}$ be two d-straight paths with $p \in \mathcal{P}_{xy}$ and $q \in \mathcal{P}_{xz}$. We denote the last intersection point of $\mathcal{P}_{xy}$ and $\mathcal{P}_{xz}$ by x_1.

We now look at the relative position of x_1 w.r.t. p and q. If x_1 is after p on $\mathcal{P}_{xy}$ and after q on $\mathcal{P}_{xz}$, then $\mathcal{C}_d\text{-}hull(\{p,q\}) \subseteq \mathcal{C}_d\text{-}hull(\{x_1, x\}) \subseteq \mathcal{C}_d\text{-}hull(\{x,y\})$. If x_1 is between p and q, say after q on $\mathcal{P}_{xz}$ but before p on $\mathcal{P}_{xy}$, then $q \in \mathcal{C}_d\text{-}hull(\{x,y\})$ and by Lemma 6.6 we again get $\mathcal{C}_d\text{-}hull(\{p,q\}) \subseteq \mathcal{C}_d\text{-}hull(\{x,y\})$. Hence, we can assume that x_1 is before p and q.

Further, we make some observations about the intersection points of $\mathcal{P}_{xy}$, $\mathcal{P}_{xz}$ and $\mathcal{P}_{yz}$. Let y_1 be the last intersection point of $\mathcal{P}_{xy}|_{yx}$ and $\mathcal{P}_{yz}$ and z_1 be the first intersection point of $\mathcal{P}_{xz}$ and $\mathcal{P}_{yz}$. Assume that y_1 is before x_1 on $\mathcal{P}_{xy}$ (see Figure 12). Then, we have

$$\begin{aligned}
\lambda_d(\mathcal{P}_{yz}) &= \lambda_d(\mathcal{P}_{yz}|_{yy_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1 z}) \\
&= \lambda_d(\mathcal{P}_{xy}|_{yx_1}) + \lambda_d(\mathcal{P}_{xy}|_{x_1 y_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1 z}) \\
&\geq \lambda_d(\mathcal{P}_{xy}|_{yx_1}) + \lambda_d(\mathcal{P}_{xz}|_{x_1 z}).
\end{aligned}$$

Hence, $\lambda_d(\mathcal{P}_{xz}|_{x_1 z}) = \lambda_d(\mathcal{P}_{xy}|_{x_1 y_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1 z})$. Since $\lambda_d(\mathcal{P}_{xz}|_{xx_1}) = \lambda_d(\mathcal{P}_{xy}|_{xy_1} * \mathcal{P}_{xy}|_{y_1 x_1})$, we have

$$\begin{aligned}
\lambda_d(\mathcal{P}_{xz}) &= \lambda_d(\mathcal{P}_{xy}|_{xy_1}) + \lambda_d(\mathcal{P}_{xy}|_{y_1 x_1}) + \lambda_d(\mathcal{P}_{xz}|_{x_1 z}) \\
&= \lambda_d(\mathcal{P}_{xy}|_{xy_1}) + \lambda_d(\mathcal{P}_{xy}|_{y_1 x_1}) + \lambda_d(\mathcal{P}_{xy}|_{x_1 y_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1 z}) \\
&\geq \lambda_d(\mathcal{P}_{xz}) + 2\lambda_d(\mathcal{P}_{xy}|_{x_1 y_1})
\end{aligned}$$

Therefore, $x_1 = y_1$. A similar argument shows that z_1 cannot be before x_1 on $\mathcal{P}_{xz}$.

Now assume that y_1 is after z_1 on $\mathcal{P}_{yz}$. We have

$$\lambda_d(\mathcal{P}_{xy}|_{xy_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1 z}) \geq \lambda_d(\mathcal{P}_{xz})$$

$$
\begin{aligned}
&= \lambda_d(\mathcal{P}_{xz}|_{zz_1}) + \lambda_d(\mathcal{P}_{xz}|_{z_1x}) \\
&= \lambda_d(\mathcal{P}_{yz}|_{zy_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1z_1}) + \lambda_d(\mathcal{P}_{xz}|_{z_1x}) \\
&\geq \lambda_d(\mathcal{P}_{yz}|_{zy_1}) + \lambda_d(\mathcal{P}_{xy}|_{y_1x}).
\end{aligned}
$$

Hence, all the inequalties above are equations. Since $\lambda_d(\mathcal{P}_{xz}|_{zz_1}) = \lambda_d(\mathcal{P}_{yz}|_{zz_1})$, we have $\lambda_d(\mathcal{P}_{xy}|_{xy_1}) = \lambda_d(\mathcal{P}_{xz}|_{xz_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1z_1})$. The above argument is symmetric in z and y and z_1 and y_1. Thus, we immediately conclude, that $y_1 = z_1$ and as above we also have $x_1 = y_1$.

We will now show that if $x_1 = y_1 = z_1$, then $\mathcal{P}_{pq} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$. We already observed that we can assume that p and q are after x_1. Hence, they belong to $\mathcal{C}_d\text{-}hull(\{y,z\})$ and $\mathcal{C}_d\text{-}hull(\{p,q\}) \subseteq \mathcal{C}_d\text{-}hull(\{y,z\})$ by Lemma 6.6 which yields the claimed inclusion.

Hence, from now on we can assume that y_1 is after x_1 on $\mathcal{P}_{xy}$ and z_1 is after x_1 on $\mathcal{P}_{xz}$ and that y_1 is before z_1 on $\mathcal{P}_{yz}$. A consequence of the above observations is that we further can assume that $\mathcal{C} := \mathcal{P}_{xy}|_{x_1y_1} * \mathcal{P}_{yz}|_{y_1z_1} * \mathcal{P}_{xz}|_{z_1x_1}$ is a Jordan curve since $\mathcal{P}_{xz}$ does not intersect $\mathcal{P}_{xy}$ after x_1 and $\mathcal{P}_{yz}$ does not intersect $\mathcal{P}_{xy}$ before y_1. Similar arguments hold for $\mathcal{P}_{xz}|_{z_1x_1}$ and $\mathcal{P}_{yz}|_{y_1z_1}$.

It will be convenient to work with x_1 instead of x. Hence, we have to show that it suffices to consider $\mathcal{C}_d\text{-}join(x_1, \mathcal{C}_d\text{-}hull(\{y,z\}))$ instead of $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$. This is, of course, implied by the containment $\mathcal{C}_d\text{-}join(x_1, \mathcal{C}_d\text{-}hull(\{y,z\})) \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ which we will prove subsequently. We first make a few statements about the location of x and $\mathcal{P}_{xy}|_{xx_1}$ w.r.t. $\mathcal{C}$.

Lemma 6.10 *If the point x is in the interior of $\mathcal{C}$, then $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ is d-convex.*

Proof: If x is in the interior of $\mathcal{C}$, then there exists an intersection point s of the part of $\overline{\mathcal{P}}_{xy}$ before x with $\mathcal{P}_{yz}|_{y_1z_1}$ or $\mathcal{P}_{xz}|_{z_1x_1}$ since $\overline{\mathcal{P}}_{xy}$ cannot be contained in $\mathcal{C}$. The first case implies that $x \in \mathcal{C}_d\text{-}hull(\{y,z\})$ and, thus, $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\})) = \mathcal{C}_d\text{-}hull(\{y,z\})$ which is, clearly, convex. On the other hand, the latter case yields that

$$
\begin{aligned}
\lambda_d(\mathcal{P}_{xz}) &= \lambda_d(\mathcal{P}_{xz}|_{xx_1}) + \lambda_d(\mathcal{P}_{xz}|_{x_1s}) + \lambda_d(\mathcal{P}_{xz}|_{sz}) \\
&= \lambda_d(\mathcal{P}_{xy}|_{xx_1}) + \lambda_d(\overline{\mathcal{P}}_{xy}|_{x_1s}) + \lambda_d(\mathcal{P}_{xz}|_{sz}) \\
&= \lambda_d(\mathcal{P}_{xy}|_{xx_1}) + \lambda_d(\mathcal{P}_{xy}|_{x_1x}) + \lambda_d(\overline{\mathcal{P}}_{xy}|_{xs}) + \lambda_d(\mathcal{P}_{xz}|_{sz}) \\
&\geq 2\lambda_d(\mathcal{P}_{xy}|_{x_1x}) + \lambda_d(\mathcal{P}_{xz})
\end{aligned}
$$

and $\lambda_d(\overline{\mathcal{P}}_{xy}|_{x_1x}) = 0$ which contradicts $x \neq x_1$ since x is in the interior of $\mathcal{C}$. □

So let x be in the exterior of $\mathcal{C}$. If there is an intersection point s of $\mathcal{P}_{xy}|_{xx_1}$ with $\mathcal{P}_{yz}$, then $\lambda_d(\mathcal{P}_{xy}|_{sy}) = \lambda_d(\mathcal{P}_{yz}|_{sy})$, and, hence, $x_1 \in \mathcal{C}_d\text{-}hull(\{y,z\})$ which implies $\mathcal{C}_d\text{-}join(x_1, \mathcal{C}_d\text{-}hull(\{y,z\})) = \mathcal{C}_d\text{-}hull(\{y,z\})$. Thus, we can assume that there is no intersection point of $\mathcal{P}_{xy}|_{xx_1}$ with $\mathcal{P}_{yz}$. Since $\mathcal{P}_{xy}|_{xx_1} \subseteq \mathcal{C}_d\text{-}hull(\{x,z\})$, there is also no intersection point of $\mathcal{P}_{xy}|_{xx_1}$ with $\mathcal{P}_{xz}|_{x_1z}$. Hence, we can assume that $\mathcal{P}_{xy}|_{xx_1}$ lies completely in $ext(\mathcal{C})$.

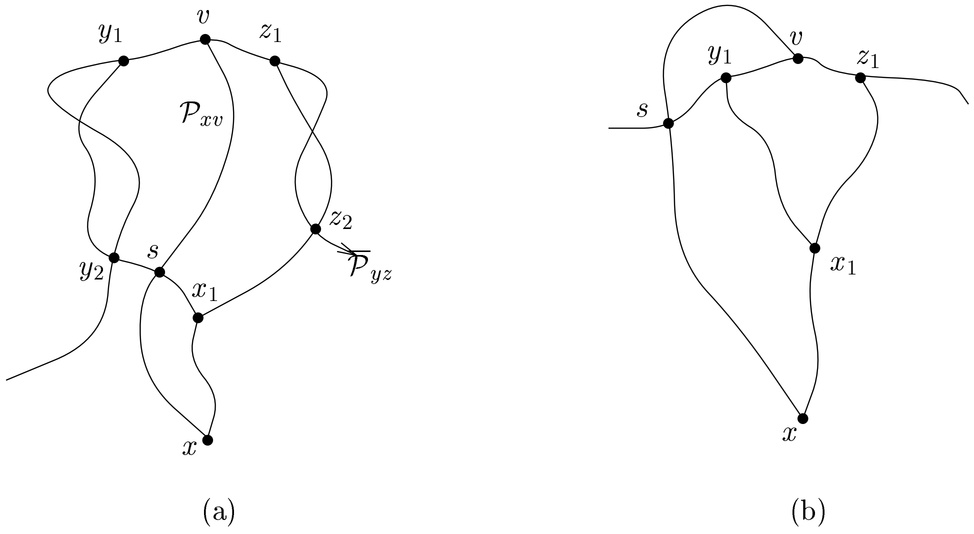


Figure 13: *$\mathcal{P}_{xv}$ intersects either $\mathcal{P}_{xy} * \mathcal{P}_{xz}$ or $\overline{\mathcal{P}}_{yz}$.*

Lemma 6.11 *With x, y, z, and x_1 chosen as above we have the following containment.*

$$\mathcal{C}_d\text{-}join(x_1, \mathcal{C}_d\text{-}hull(\{y,z\})) \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\})).$$

Proof: The claim is trivial if x_1 equals x. Hence, we assume that x_1 occurs after x on $\mathcal{P}_{xy}$. By the above observations we only have to show that $\mathcal{C}_d\text{-}hull(\{x_1, v\}) \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$, for all points v in $\mathcal{P}_{yz}|_{y_1 z_1}$. For if $v \in \mathcal{P}_{yz}|_{yy_1} \subseteq \mathcal{C}_d\text{-}hull(\{x,y\})$, then $x_1, v \in \mathcal{C}_d\text{-}hull(\{x,y\})$ and $\mathcal{C}_d\text{-}hull(\{x_1,v\}) \subseteq \mathcal{C}_d\text{-}hull(\{x,y\}) \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$. The same, of course, holds if $v \in \mathcal{P}_{yz}|_{z_1 z}$. Therefore, let $\mathcal{P}_{x_1 v}$ be a d-straight path from x_1 to some point $v \in \mathcal{C}_d\text{-}hull(\{y_1, z_1\})$.

Let y_2 be the first intersection point of $\overline{\mathcal{P}}_{yz}$ with $\mathcal{P}_{xy}|_{x_1 y}$ on $\mathcal{P}_{xy}|_{x_1 y}$ and z_2 be the first intersection point of $\overline{\mathcal{P}}_{yz}$ with $\mathcal{P}_{xz}|_{x_1 z}$ on $\mathcal{P}_{xz}|_{x_1 z}$ (see Figure 13a). By the above argument we can assume that y_2 is before y_1 and z_2 is after z_1 on $\mathcal{P}_{yz}$. We denote the part of $\overline{\mathcal{P}}_{yz}$ before y_2 by $\overline{\mathcal{P}}_{yz}|_{-\infty y_2}$ and the part of $\overline{\mathcal{P}}_{yz}$ after z_2 by $\overline{\mathcal{P}}_{yz}|_{z_2 \infty}$. Obviously, $\overline{\mathcal{P}}_{yz}|_{-\infty z_2}$ and $\overline{\mathcal{P}}_{yz}|_{z_2 \infty}$ are contained in $ext(\mathcal{C})$.

Consider the curve $\mathcal{C}_1 = \overline{\mathcal{P}}_{yz}|_{-\infty y_2} * \mathcal{P}_{xy}|_{y_2 x_1} * \mathcal{P}_{xz}|_{x_1 z_2} * \overline{\mathcal{P}}_{yz}|_{z_2 \infty}$. Since $\mathcal{C}_1$ is biinfinite and simple, it splits $\mathbf{X}$ into at least two components. $int(\mathcal{C})$ does not intersect $\mathcal{C}_1$ and, hence, there is a component, say $\mathbf{Q}_1$, that contains $int(\mathcal{C})$. If we consider a sufficiently small neighbourhood N of x_1, we have that $N \cap int(\mathcal{C}) = N \cap \mathbf{Q}_1$ since there is a neighbourhood of x_1 in which $\mathcal{C}$ and $\mathcal{C}_1$ are identical.

If $\mathcal{P}_{xy}|_{xx_1}$ intersects $\mathcal{C}_1$ in a point t apart from x_1, then $\mathcal{P}_{xy}|_{xx_1}$ either intersects $\overline{\mathcal{P}}_{yz}|_{-\infty y_2}$ or $\overline{\mathcal{P}}_{yz}|_{z_2 \infty}$. W.l.o.g. assume that $t \in \mathcal{P}_{xy}|_{xx_1} \cap \overline{\mathcal{P}}_{yz}|_{z_2 \infty}$. Then, $\lambda_d(\mathcal{P}_{xy}|_{ty}) = \lambda_d(\overline{\mathcal{P}}_{yz}|_{ty})$ and, hence, $z_1 \in \mathcal{C}_d\text{-}hull(\{x,y\})$ and $\mathcal{P}_{x_1 v} \subseteq \mathcal{C}_d\text{-}hull(\{x,y\})$. Thus, we can assume that $\overline{\mathcal{P}}_{yz} \setminus \mathcal{P}_{yz}$ does not intersect $\mathcal{P}_{xx_1}$. Therefore, $\mathcal{P}_{xy}|_{xx_1}$ does not cross $\mathcal{C}_1$. On the other hand, $\mathcal{P}_{xy}|_{xx_1}$

does not intersect $int(\mathcal{C})$. Hence, x is in a component $\mathbf{Q}_2$ which is different from $\mathbf{Q}_1$. Now let $\mathcal{P}_{xv}$ be a d-straight path from x to v. Since $v \notin \mathcal{C}_1$ and v can be connected to any point in $int(\mathcal{C})$ by a path that does not cross $\mathcal{C}_1$, v belongs to $\mathbf{Q}_1$, and $\mathcal{P}_{xv}$ crosses $\mathcal{C}_1$. Since $\mathcal{C}_1 \subseteq \mathcal{P}_{xy} \cup \mathcal{P}_{xz} \cup \overline{\mathcal{P}}_{yz} \setminus \mathcal{P}_{yz}$, we have to consider the following cases.

Case 1 $\mathcal{P}_{xv}$ intersects $\mathcal{P}_{xy}$ in s (see Figure 13a).
We then get

$$\begin{aligned}
\lambda_d(\mathcal{P}_{xv}) &= \lambda_d(\mathcal{P}_{xv}|_{xs}) + \lambda_d(\mathcal{P}_{xv}|_{sv}) \\
&= \lambda_d(\mathcal{P}_{xy}|_{xs}) + \lambda_d(\mathcal{P}_{xv}|_{sv}) \\
&= \lambda_d(\mathcal{P}_{xy}|_{xx_1}) + \lambda_d(\mathcal{P}_{xy}|_{x_1 s}) + \lambda_d(\mathcal{P}_{xv}|_{sv}) \\
&\geq \lambda_d(\mathcal{P}_{xy}|_{xx_1}) + \lambda_d(\mathcal{P}_{x_1 v})
\end{aligned}$$

and $\mathcal{P}_{xy}|_{xx_1} * \mathcal{P}_{x_1 v}$ is shortest and, hence, d-straight path from x to v. Therefore, we have $\mathcal{P}_{x_1 v} \subseteq \mathcal{C}_d\text{-}hull(\{x, v\})$.

Case 2 $\mathcal{P}_{xv}$ intersects $\mathcal{P}_{xz}$ in s (see Figure 13b).
A similar argument to the case that $\mathcal{P}_{xv}$ intersects $\mathcal{P}_{xy}$ gives the desired result.

Case 3 $\mathcal{P}_{xv}$ intersects $\overline{\mathcal{P}}_{yz}$ before y in s.
Then, we have

$$\begin{aligned}
\lambda_d(\mathcal{P}_{xv}) &= \lambda_d(\mathcal{P}_{xv}|_{xs}) + \lambda_d(\mathcal{P}_{xv}|_{sv}) \\
&= \lambda_d(\mathcal{P}_{xv}|_{xs}) + \lambda_d(\overline{\mathcal{P}}_{yz}|_{sv}) \\
&= \lambda_d(\mathcal{P}_{xv}|_{xs}) + \lambda_d(\overline{\mathcal{P}}_{yz}|_{sy_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1 v}) \\
&\geq \lambda_d(\mathcal{P}_{xy}|_{xy_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1 v}) \\
&= \lambda_d(\mathcal{P}_{xy}|_{xx_1}) + \lambda_d(\mathcal{P}_{xy}|_{x_1 y_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1 v}) \\
&\geq \lambda_d(\mathcal{P}_{xy}|_{xx_1}) + \lambda_d(\mathcal{P}_{x_1 v})
\end{aligned}$$

and $\mathcal{P}_{x_1 v} \subseteq \mathcal{C}_d\text{-}hull(\{x, v\})$.

Case 4 $\mathcal{P}_{xv}$ intersects $\overline{\mathcal{P}}_{yz}$ after z in s.
A similar argument to the case that $\mathcal{P}_{xv}$ intersects $\overline{\mathcal{P}}_{yz}$ before y yields the desired result.

So we have in any case that $\mathcal{P}_{x_1 v} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$. □

Because of the above lemma we can assume that $x = x_1$. Before we continue we show that all the points in $int(\mathcal{C})$ belong to $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$.

Lemma 6.12 *If $v \in int(\mathcal{C})$, then $v \in \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$.*

Proof: Since v is contained in a d-triangle, there exists a d-straight path $\mathcal{P}_{xv}$ from x to v. $\overline{\mathcal{P}}_{xv}$ cannot be contained in $int(\mathcal{C})$, hence, there is an intersection point s of $\overline{\mathcal{P}}_{xv}$ with $\mathcal{C}$ after v. If s belongs to $\mathcal{P}_{xy}$, then $\mathcal{P}_{xv} \subseteq \mathcal{P}_{xy}$ and $v \in \mathcal{C}_d\text{-}hull(\{x, y\})$. The analog holds if $s \in \mathcal{P}_{xz}$. If $s \in \mathcal{P}_{yz}$, then $v \in \mathcal{C}_d\text{-}hull(\{x, s\}) \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$. □

We now consider the path connecting p and q. Let p_1 be the last intersection point of $\mathcal{P}_{pq}$ with $\mathcal{P}_{xy}$ on $\mathcal{P}_{pq}$ and q_1 be the first intersection point of $\mathcal{P}_{pq}$ with $\mathcal{P}_{xz}$ on $\mathcal{P}_{pq}$. Clearly, $p, p_1 \in \mathcal{C}_d\text{-}hull(\{x, y\})$ and, thus, $\mathcal{P}_{pq}|_{pp_1} \subseteq \mathcal{C}_d\text{-}hull(\{x, y\})$. Similarly, we have $\mathcal{P}_{pq}|_{q_1q} \subseteq \mathcal{C}_d\text{-}hull(\{x, z\})$. If p_1 occurs after q_1 on $\mathcal{P}_{pq}$, then $\mathcal{P}_{pq} \subseteq \mathcal{C}_d\text{-}hull(\{x, y\}) \cup \mathcal{C}_d\text{-}hull(\{x, z\})$. So, w.l.o.g. we can assume that $p = p_1$ and $q = q_1$ and that $\mathcal{P}_{pq}$ does not intersect $\mathcal{P}_{xy}$ and $\mathcal{P}_{xz}$.

We summarize the results of this section as follows. W.l.o.g. we can assume that

(i) $\mathcal{P}_{xy}$ and $\mathcal{P}_{xz}$ do not intersect (except in x),

(ii) $\mathcal{C} = \mathcal{P}_{xy}|_{xy_1} * \mathcal{P}_{yz}|_{y_1z_1} * \mathcal{P}_{xz}|_{z_1x}$ is a Jordan curve,

(iii) $int(\mathcal{C}) \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$, and

(iv) $\mathcal{P}_{pq}$ does neither intersect $\mathcal{P}_{xy}$ nor $\mathcal{P}_{xz}$ (expect in p and q).

6.9.2 The Four Cases of the Proof

We classify the possible situtations that can arise according to the relative position of y_1 and p and z_1 and q. We have the following four cases.

1. p is after y_1 on $\mathcal{P}_{xy}$ and q is after z_1 on $\mathcal{P}_{xz}$.
2. p is after y_1 and q is before z_1.
3. p is before y_1 and q is after z_1.
4. p and q are on $\mathcal{C}$.

The first case is the simplest of all. If p is after y_1 and q is after z_1, then $p \in \mathcal{C}_d\text{-}hull(\{y, y_1\}) \subseteq \mathcal{C}_d\text{-}hull(\{y, z\})$ and $q \in \mathcal{C}_d\text{-}hull(\{z_1, z\}) \subseteq \mathcal{C}_d\text{-}hull(\{y, z\})$. Lemma 6.6 yields that $\mathcal{C}_d\text{-}hull(\{p, q\}) \subseteq \mathcal{C}_d\text{-}hull(\{y, z\})$.

Since Cases 2 and 3 are symmetric w.r.t. y_1 and z_1 and p and q, it suffices to consider one of them. So let p be after y_1 and q before z_1. By Property (i) of the last section $\mathcal{P}_{pq}$ does not intersect $\mathcal{P}_{xy}$ and $\mathcal{P}_{xz}$. So we have the following possiblities.

(a) $\mathcal{P}_{pq}$ is completely contained in $int(\mathcal{C})$.

(b) $\mathcal{P}_{pq}$ is completely contained in $ext(\mathcal{C})$.

(c) $\mathcal{P}_{pq}$ intersects $\mathcal{P}_{yz}$.

Since Possibility 1 immediately implies that $\mathcal{P}_{pq}$ is contained in $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$ by Lemma 6.12, we will not consider this case anymore.

In order to treat Possibilities 2 and 3, we need a few more preparations. In Section 6.8 we constructed intersection points of $\mathcal{P}_{pq}$ with the extensions of $\mathcal{P}_{xy}$, $\mathcal{P}_{xz}$, and $\mathcal{P}_{yz}$. We now show that if $\mathcal{P}_{pq}$ is a d-straight path, then the existence of certain intersection points implies the d-convexity of $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$.

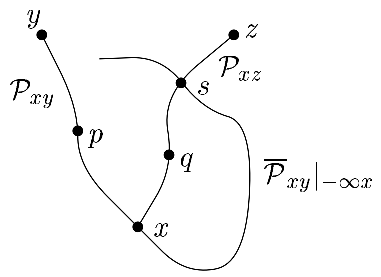


Figure 14: *The part of $\overline{\mathcal{P}}_{xy}$ before x intersects $\mathcal{P}_{xz}$ between q and z.*

Lemma 6.13 *If the part of $\overline{\mathcal{P}}_{xy}$ before x intersects $\mathcal{P}_{xz}$ between q and z, then $\mathcal{P}_{pq}$ belongs to $\mathcal{C}_d$-join$(x, \mathcal{C}_d$-hull$(\{y, z\}))$.*

Proof: Let s be the intersection point of $\overline{\mathcal{P}}_{xy}|_{-\infty x}$ with $\mathcal{P}_{xz}|_{qz}$. We then have $x \in \mathcal{C}_d$-*hull*$(\{y, s\})$ and $s \in \mathcal{C}_d$-*hull*$(\{x, z\})$. By the stiffness of d we conclude that $\mathcal{C}_d$-*hull*$(\{y, s\}) \subseteq \mathcal{C}_d$-*join*$(x, \mathcal{C}_d$-*hull*$(\{y, z\}))$. Since, $p, q \in \mathcal{C}_d$-*hull*$(\{y, s\})$, the claim follows immediately (see Figure 14). $\square$

Lemma 6.14 *If $\mathcal{P}_{pq}$ intersects both $\overline{\mathcal{P}}_{xy}$ in some point r and $\overline{\mathcal{P}}_{xz}$ in some point s such that r occurs before x on $\overline{\mathcal{P}}_{xy}$ and s occurs before x on $\overline{\mathcal{P}}_{xz}$, then $\mathcal{C}_d$-join$(x, \mathcal{C}_d$-hull$(\{y, z\}))$ contains $\mathcal{P}_{pq}$.*

Proof: Let r be a intersection point of $\mathcal{P}_{pq}$ and $\overline{\mathcal{P}}_{xy}$ before x and s a intersection point of $\mathcal{P}_{pq}$ and $\overline{\mathcal{P}}_{xz}$ before x (see Figure 15).
First assume that r is before s on $\mathcal{P}_{pq}$. We have

$$\lambda_d(\mathcal{P}_{xy}|_{px}) + \lambda_d(\overline{\mathcal{P}}_{xy}|_{xr}) = \lambda_d(\mathcal{P}_{pq}|_{pr})$$

and

$$\lambda_d(\mathcal{P}_{xz}|_{qx}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{xs}) = \lambda_d(\mathcal{P}_{pq}|_{qs})$$

This implies

$$\begin{aligned}
\lambda_d(\mathcal{P}_{xy}|_{px}) + \lambda_d(\mathcal{P}_{xz}|_{xq}) &\leq \lambda_d(\mathcal{P}_{xy}|_{px}) + \lambda_d(\overline{\mathcal{P}}_{xy}|_{xr}) + \lambda_d(\mathcal{P}_{xz}|_{xq}) + \\
&\quad\ \lambda_d(\overline{\mathcal{P}}_{xz}|_{xs}) \\
&= \lambda_d(\mathcal{P}_{pq}|_{pr}) + \lambda_d(\mathcal{P}_{pq}|_{sq}) \\
&< \lambda_d(\mathcal{P}_{pq}|_{pr}) + \lambda_d(\mathcal{P}_{pq}|_{rs}) + \lambda_d(\mathcal{P}_{pq}|_{sq}) \\
&\leq \lambda_d(\mathcal{P}_{pq}).
\end{aligned}$$

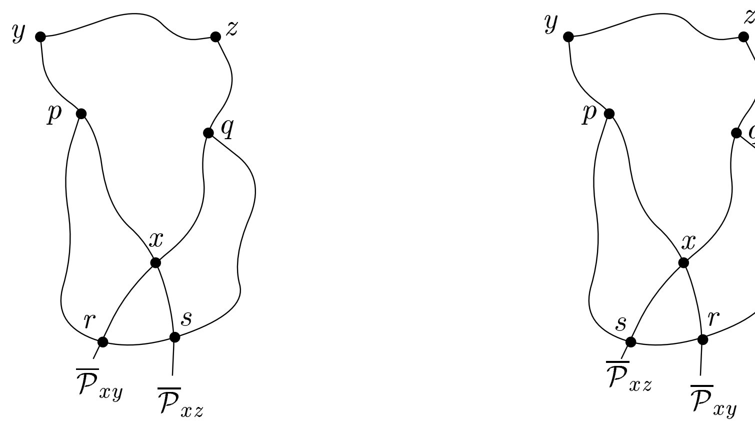


Figure 15: *$\mathcal{P}_{pq}$ intersects $\overline{\mathcal{P}}_{xy}$ and $\overline{\mathcal{P}}_{xz}$ before x.*

Hence, $\lambda_d(\overline{\mathcal{P}}_{xy}|_{xr}) = \lambda_d(\overline{\mathcal{P}}_{xz}|_{xs}) = 0$ and $\mathcal{P}_{pq}|_{rs} = x$ and $\mathcal{P}_{pq}|_{pr} \subseteq \mathcal{C}_d$-*hull*$(\{x, y\})$ and $\mathcal{P}_{pq}|_{sq} \subseteq \mathcal{C}_d$-*hull*$(\{x, z\})$.

Now suppose that s is before r on $\mathcal{P}_{pq}$. In this case we have

$$\begin{aligned} \lambda_d(\mathcal{P}_{pq}) &= \lambda_d(\mathcal{P}_{pq}|_{pr}) + \lambda_d(\mathcal{P}_{pq}|_{rq}) \\ &= \lambda_d(\mathcal{P}_{xy}|_{px}) + \lambda_d(\overline{\mathcal{P}}_{xy}|_{xr}) + \lambda_d(\mathcal{P}_{pq}|_{rq}) \\ &\geq \lambda_d(\mathcal{P}_{xy}|_{px}) + \lambda_d(\mathcal{P}_{xz}|_{xq}). \end{aligned}$$

Hence, $\lambda_d(\overline{\mathcal{P}}_{xy}|_{xr}) + \lambda_d(\mathcal{P}_{pq}|_{rq}) = \lambda_d(\mathcal{P}_{xz}|_{xq})$ and in the same way we get $\lambda_d(\overline{\mathcal{P}}_{xz}|_{xs}) + \lambda_d(\mathcal{P}_{pq}|_{sp}) = \lambda_d(\mathcal{P}_{xy}|_{xp})$. Hence, $s \in \mathcal{C}_d$-*hull*$(\{x, y\})$ and $r \in \mathcal{C}_d$-*hull*$(\{x, z\})$. Furthermore,

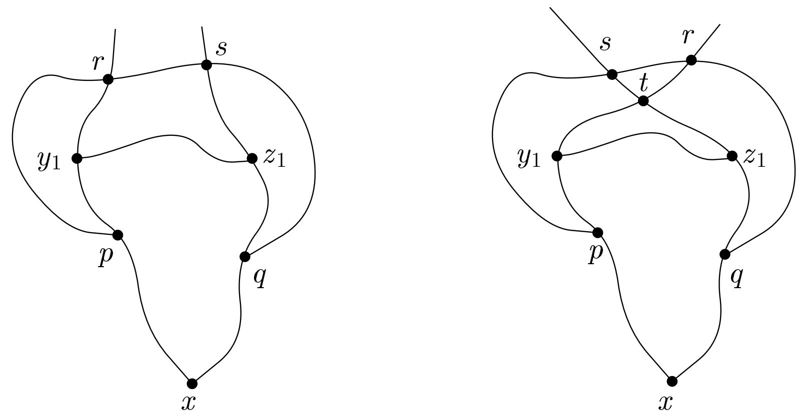
$$\begin{aligned} &\lambda_d(\mathcal{P}_{xy}|_{px}) + \lambda_d(\overline{\mathcal{P}}_{xy}|_{xr}) + \lambda_d(\mathcal{P}_{xz}|_{qx}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{xs}) \\ &\quad = \lambda_d(\mathcal{P}_{pq}|_{pr}) + \lambda_d(\mathcal{P}_{pq}|_{sq}) \\ &\quad = \lambda_d(\mathcal{P}_{pq}) + \lambda_d(\mathcal{P}_{pq}|_{sr}). \end{aligned}$$

Therefore, $\lambda_d(\overline{\mathcal{P}}_{xy}|_{xr}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{xs}) = \lambda_d(\mathcal{P}_{pq}|_{sr})$ and $x \in \mathcal{C}_d$-*hull*$(\{s, r\}) \subseteq \mathcal{C}_d$-*hull*$(\{y, r\})$.

Thus, we have $x \in \mathcal{C}_d$-*hull*$(\{y, r\})$ and $r \in \mathcal{C}_d$-*hull*$(\{x, z\})$. The stiffness of d yields that $\mathcal{C}_d$-*hull*$(\{y, r\}) \subseteq \mathcal{C}_d$-*join*$(x, \mathcal{C}_d$-*hull*$(\{y, z\}))$. Since $p \in \mathcal{C}_d$-*hull*$(\{y, r\})$, we have $\mathcal{P}_{pq}|_{pr} \subseteq \mathcal{C}_d$-*join*$(x, \mathcal{C}_d$-*hull*$(\{y, z\}))$. Further, $r, q \in \mathcal{C}_d$-*hull*$(\{x, z\})$ and, therefore, $\mathcal{P}_{pq}|_{rq} \subseteq \mathcal{C}_d$-*join*$(x, \mathcal{C}_d$-*hull*$(\{y, z\}))$. □

The above proof remains valid if we consider y_1 and $\overline{\mathcal{P}}_{xy}$ and $\overline{\mathcal{P}}_{yz}$ or z_1 and $\overline{\mathcal{P}}_{xz}$ and $\overline{\mathcal{P}}_{yz}$ instead of x and $\overline{\mathcal{P}}_{xy}$ and $\overline{\mathcal{P}}_{xz}$. For the following lemma we define $\mathcal{J}$ as the curve $\mathcal{P}_{pq} * \mathcal{P}_{xz}|_{qx} * \mathcal{P}_{xy}|_{xp}$. Note that if $\mathcal{P}_{pq}$ is contained in $ext(\mathcal{C})$, then $\mathcal{J}$ is a Jordan curve.

Lemma 6.15 *If $\mathcal{P}_{pq}$ is contained in $ext(\mathcal{C})$, $int(\mathcal{J})$ contains z_1, and $\mathcal{P}_{pq}$ intersects both $\overline{\mathcal{P}}_{xy}$ in some point r and $\overline{\mathcal{P}}_{xz}$ in some point s such that r occurs after y_1 on $\overline{\mathcal{P}}_{xy}$ and s occurs after z_1 on $\overline{\mathcal{P}}_{xz}$, then $\mathcal{C}_d$-join$(x, \mathcal{C}_d$-hull$(\{y, z\}))$ contains $\mathcal{P}_{pq}$.*



(a) (b)

Figure 16: *$\mathcal{P}_{pq}$ intersects $\overline{\mathcal{P}}_{xy}$ after y and $\overline{\mathcal{P}}_{xz}$ after z.*

Proof: Let s be the first intersection point of $\overline{\mathcal{P}}_{xz}|_{z_1\infty}$ and $\mathcal{P}_{pq}$ on $\overline{\mathcal{P}}_{xz}|_{z_1\infty}$ and r the first intersection point of $\overline{\mathcal{P}}_{xy}|_{y_1\infty}$ and $\mathcal{P}_{pq}$ on $\overline{\mathcal{P}}_{xy}|_{y_1\infty}$. If r is before s on $\mathcal{P}_{pq}$, we have that (see Figure 16a)

$$
\begin{aligned}
\lambda_d(\mathcal{P}_{pq}) &\leq \lambda_d(\overline{\mathcal{P}}_{xy}|_{py_1}) + \lambda_d(\mathcal{P}_{yz}|_{y_1z_1}) + \lambda_d(\mathcal{P}_{xz}|_{z_1q}) \\
&\leq \lambda_d(\overline{\mathcal{P}}_{xy}|_{py_1}) + \lambda_d(\overline{\mathcal{P}}_{xy}|_{y_1r}) + \lambda_d(\mathcal{P}_{pq}|_{rs}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{sz_1}) + \\
&\quad\ \lambda_d(\overline{\mathcal{P}}_{xz}|_{z_1q}) \\
&= \lambda_d(\mathcal{P}_{pq}).
\end{aligned}
$$

Hence, $\lambda_d(\overline{\mathcal{P}}_{xy}|_{y_1r}) + \lambda_d(\mathcal{P}_{pq}|_{rs}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{sz_1}) = \lambda_d(\mathcal{P}_{yz}|_{y_1z_1})$ and $\overline{\mathcal{P}}_{xy}|_{y_1r} * \mathcal{P}_{pq}|_{rs} * \overline{\mathcal{P}}_{xz}|_{sz_1} \subseteq \mathcal{C}_d\text{-}hull(\{y_1, z_1\}) \subseteq \mathcal{C}_d\text{-}hull(\{y, z\})$ and $\mathcal{P}_{pq}|_{pr} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$ and $\mathcal{P}_{pq}|_{sq} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$.

Now assume that r is after s on $\mathcal{P}_{pq}$ as shown in Figure 16b. If $\overline{\mathcal{P}}_{xz}|_{z_1s}$ intersects $\mathcal{P}_{xy}$ in a point t, then $t \in \mathcal{C}_d\text{-}hull(\{x, y\})$ and $z_1 \in \mathcal{C}_d\text{-}hull(\{x, t\})$ implies that $\mathcal{C}_d\text{-}hull(\{x, y\})$ contains p and q and, therefore, $\mathcal{P}_{pq}$. Hence, suppose this is not the case. Then, $\mathcal{J}' = \overline{\mathcal{P}}_{xz}|_{xs} * \mathcal{P}_{pq}|_{sp} * \mathcal{P}_{xy}|_{px}$ is a Jordan curve. Since $\overline{\mathcal{P}}_{xz}|_{qs}$ is a simple path that intersects $\mathcal{J}$ exactly twice and that intersects points in the interior of $\mathcal{J}$, we have that $int(\mathcal{J}') \subseteq int(\mathcal{J})$. Furthermore, since $\mathcal{P}_{xy}|_{y_1p}$ does not intersect $\overline{\mathcal{P}}_{xz}|_{qs}$ by assumption, y_1 is contained in $int(\mathcal{J}')$. r does not belong to $\mathcal{J}'$ and, hence, $\overline{\mathcal{P}}_{xy}|_{y_1r}$ intersects $\mathcal{J}'$. If $\overline{\mathcal{P}}_{xy}|_{y_1r}$ intersects $\mathcal{P}_{xz}$ in a point t, then it can be shown as above that $\mathcal{P}_{pq}$ belongs to $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$. $\overline{\mathcal{P}}_{xy}$ is simple and, therefore, $\overline{\mathcal{P}}_{xy}|_{y_1r}$ does not intersect $\mathcal{P}_{xy}|_{xp}$. Furthermore, we assume that $\overline{\mathcal{P}}_{xy}|_{y_1r}$ does not intersect $\mathcal{P}_{pq}|_{ps}$. Thus, $\overline{\mathcal{P}}_{xy}|_{y_1r}$ intersects $\overline{\mathcal{P}}_{xz}|_{zs}$ in a point t. Then, we have

$$
\lambda_d(\mathcal{P}_{pq}|_{pr}) \quad = \quad \lambda_d(\overline{\mathcal{P}}_{xy}|_{pt}) + \lambda_d(\overline{\mathcal{P}}_{xy}|_{tr})
$$

and

$$\lambda_d(\mathcal{P}_{pq}|_{qs}) \;=\; \lambda_d(\overline{\mathcal{P}}_{xz}|_{qt}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{ts}).$$

Further,

$$\begin{aligned}\lambda_d(\mathcal{P}_{pq}) &= \lambda_d(\mathcal{P}_{pq}|_{pr}) + \lambda_d(\mathcal{P}_{pq}|_{rq})\\ &= \lambda_d(\overline{\mathcal{P}}_{xy}|_{pt}) + \lambda_d(\overline{\mathcal{P}}_{xy}|_{tr}) + \lambda_d(\mathcal{P}_{pq}|_{rq})\\ &\geq \lambda_d(\overline{\mathcal{P}}_{xy}|_{pt}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{tq}) \qquad (3)\end{aligned}$$

Hence,

$$\begin{aligned}\lambda_d(\overline{\mathcal{P}}_{xy}|_{pt}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{tq}) &= \lambda_d(\mathcal{P}_{pq}|_{pr}) - \lambda_d(\overline{\mathcal{P}}_{xy}|_{tr}) + \lambda_d(\mathcal{P}_{pq}|_{qs}) - \\ \lambda_d(\overline{\mathcal{P}}_{xz}|_{ts}) &\\ &= \lambda_d(\mathcal{P}_{pq}) + \lambda_d(\mathcal{P}_{pq}|_{sr}) - \lambda_d(\overline{\mathcal{P}}_{xy}|_{tr}) - \\ &\quad\ \lambda_d(\overline{\mathcal{P}}_{xz}|_{ts})\\ \overset{(3)}{\Rightarrow} \qquad \lambda_d(\mathcal{P}_{pq}|_{sr}) &= \lambda_d(\overline{\mathcal{P}}_{xy}|_{tr}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{ts}) \qquad (4)\end{aligned}$$

since $\lambda_d(\mathcal{P}_{pq}|_{sr}) \leq \lambda_d(\mathcal{P}_{xy}|_{rt}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{ts})$. With

$$\begin{aligned}\lambda_d(\overline{\mathcal{P}}_{xy}|_{pt}) &= \lambda_d(\mathcal{P}_{pq}|_{pr}) - \lambda_d(\overline{\mathcal{P}}_{xy}|_{tr})\\ &= \lambda_d(\mathcal{P}_{pq}|_{ps}) + \lambda_d(\mathcal{P}_{pq}|_{sr}) - \lambda_d(\overline{\mathcal{P}}_{xy}|_{rt})\\ &\overset{(4)}{=} \lambda_d(\mathcal{P}_{pq}|_{ps}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{st}) \qquad (5)\end{aligned}$$

we obtain

$$\begin{aligned}\lambda_d(\overline{\mathcal{P}}_{xz}|_{xs}) &= \lambda_d(\overline{\mathcal{P}}_{xz}|_{xt}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{ts})\\ &= \lambda_d(\overline{\mathcal{P}}_{xy}|_{xt}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{ts})\\ &= \lambda_d(\overline{\mathcal{P}}_{xy}|_{xp}) + \lambda_d(\overline{\mathcal{P}}_{xy}|_{pt}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{ts})\\ &\overset{(5)}{=} \lambda_d(\overline{\mathcal{P}}_{xy}|_{xp}) + \lambda_d(\mathcal{P}_{pq}|_{ps}) + 2\lambda_d(\overline{\mathcal{P}}_{xz}|_{ts})\\ &\geq \lambda_d(\overline{\mathcal{P}}_{xz}|_{xs}) + 2\lambda_d(\overline{\mathcal{P}}_{xz}|_{ts}).\end{aligned}$$

Thus, we have $s = r = t$ and which reduces the problem to the first case with $r = s$. $\square$

6.9.3 Cases 2 and 3

We are now in a position to continue to consider Case 2. One possibility is that $\mathcal{P}_{pq}$ is completely contained in the exterior of $\mathcal{C}$. In this case we consider the curve $\mathcal{P} = \mathcal{P}_{xy}|_{xp} * \mathcal{P}_{pq} * \mathcal{P}_{xz}|_{qx}$ which is a Jordan curve according to our assumptions.

First assume that $z \in int(\mathcal{P})$ as shown in Figure 17a. Since $\overline{\mathcal{P}}_{xz}$ cannot be contained in a Jordan-curve, it intersects $\mathcal{P}$ in some point s that occurs after z on $\overline{\mathcal{P}}_{xz}$. If $\overline{\mathcal{P}}_{xz}$ does not

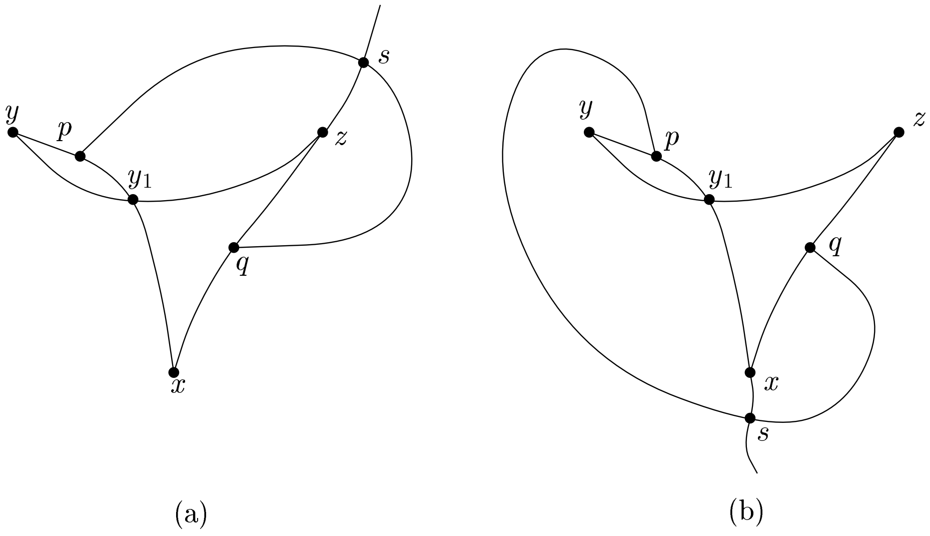


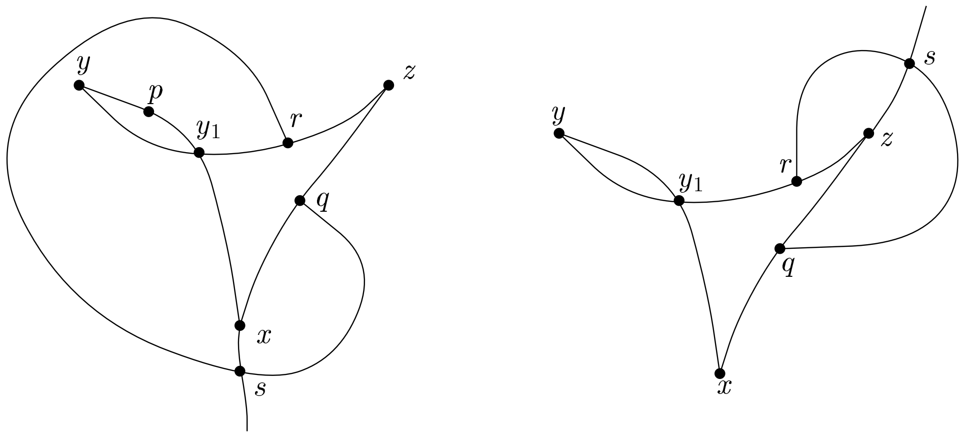
Figure 17: *Illustrating Case 2.*

intersects $\mathcal{P}_{pq}$, then $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ is d-convex by Lemma 6.7. Hence, suppose $s \in \mathcal{P}_{pq}$. Applying Lemma 6.9 yields that $\lambda_d(\mathcal{P}_{pq}|_{ps}) + \lambda_d(\overline{\mathcal{P}}_{xz}|_{sz}) = \lambda_d(\mathcal{P}_{xy}|_{py_1} * \mathcal{P}_{yz}|_{y_1z})$. Hence, $s \in \mathcal{C}_d\text{-}hull(\{y,z\})$ and $\mathcal{P}_{pq}|_{ps} \subseteq \mathcal{C}_d\text{-}hull(\{y,z\})$. Further, $\mathcal{P}_{pq}|_{qs} \subseteq \mathcal{C}_d\text{-}hull(\{x,s\})$ and, thus, $\mathcal{P}_{pq}|_{qs} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$.

Now assume that $z \in ext(\mathcal{P})$ which is illustrated in Figure 17b. By Lemma 6.8 $\overline{\mathcal{P}}_{xy}$ intersects $\mathcal{P}_{pq} * \mathcal{P}_{xz}|_{qz}$ in some point s that occurs before x on $\overline{\mathcal{P}}_{xy}$ or $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ contains $\mathcal{P}_{pq}$. If s belongs to $\mathcal{P}_{xz}|_{qz}$, then Lemma 6.13 yields that $\mathcal{P}_{pq}$ belongs to $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$. Hence, we can assume that s belongs to $\mathcal{P}_{pq}$. By Lemma 6.9 we have that $\lambda_d(\mathcal{P}_{xz}|_{xq}) = \lambda_d(\overline{\mathcal{P}}_{xy}|_{xs}) + \lambda_d(\mathcal{P}_{pq}|_{sq})$. This immediately gives $\mathcal{P}_{pq}|_{sq} \subseteq \mathcal{C}_d\text{-}hull(\{x,z\})$. Furthermore, we have $x \in \mathcal{C}_d\text{-}hull(\{y,s\})$ and $s \in \mathcal{C}_d\text{-}hull(\{x,z\})$. By the stiffness of d we conclude that $\mathcal{C}_d\text{-}hull(\{y,s\}) \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ which, in particular, yields $\mathcal{P}_{pq}|_{ps} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ and, thus, $\mathcal{P}_{pq}$ is contained in $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$. Since we never made use of the fact that p belongs to $\mathcal{C}_d\text{-}hull(\{y,z\})$, the same considerations can also be used in the last case, Case 4. For further reference we state the result as a lemma.

Lemma 6.16 *If p and q are two points in $\mathcal{P}_{xy}$ resp. $\mathcal{P}_{xz}$ and $\mathcal{P}_{pq}$ is a d-straight path connecting them such that z is in the exterior of the Jordan curve $\mathcal{P} = \mathcal{P}_{pq} * \mathcal{P}_{xz}|_{qx} * \mathcal{P}_{xy}|_{xq}$, then $\mathcal{P}_{pq}$ is contained in $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$.*

So the above considerations can be summarized that if $\mathcal{P}_{pq}$ is contained in $ext(\mathcal{C})$, then $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ contains $\mathcal{P}_{pq}$. Since we have already considered the case that $\mathcal{P}_{pq}$ is contained in $int(\mathcal{C})$, the only remaining case is that $\mathcal{P}_{pq}$ intersects $\mathcal{P}_{yz}$. Let r be the last intersection point of $\mathcal{P}_{pq}$ and $\mathcal{P}_{yz}$ on $\mathcal{P}_{pq}$. Clearly, $\mathcal{P}_{pq}|_{pr} \subseteq \mathcal{C}_d\text{-}hull(\{y,z\})$ since



(a) (b)

Figure 18: *$\mathcal{P}_{pq}$ crosses $\mathcal{P}_{yz}$.*

$p \in \mathcal{C}_d\text{-}hull(\{y,z\})$. Now, if $\mathcal{P}_1 = \mathcal{P}_{pq}|_{rq}$ is contained in $int(\mathcal{C})$, then $\mathcal{P}_{pq}$ is contained in $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ by Lemma 6.12. Hence, assume that $\mathcal{P}_1 \subseteq ext(\mathcal{C})$. We have to show that $\mathcal{P}_1$ belongs to $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$. Let $\mathcal{P} = \mathcal{P}_1 * \mathcal{P}_{xz}|_{qz_1} * \mathcal{P}_{yz}|_{z_1r}$. There are again two cases. If x belongs to $int(\mathcal{P})$ as shown in Figure 18, then we claim that we can assume that $\overline{\mathcal{P}}_{xy}$ and $\overline{\mathcal{P}}_{xz}$ intersect $\mathcal{P}_{pq}$ before x. To see this consider first $\overline{\mathcal{P}}_{xy}|_{-\infty x}$. If $\overline{\mathcal{P}}_{xy}|_{-\infty x}$ intersects $\mathcal{P}_{xz}|_{qz_1}$, then $\mathcal{P}_{pq}$ belongs to $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ by Lemma 6.13. If $\overline{\mathcal{P}}_{xy}|_{-\infty x}$ intersects $\mathcal{P}_{yz}|_{rz_1}$ in a point s, then we have $s \in \mathcal{C}_d\text{-}hull(\{y,z\})$ and $x \in \mathcal{C}_d\text{-}hull(\{y,s\})$ and, hence, $x \in \mathcal{C}_d\text{-}hull(\{y,z\})$ and $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ is convex. Therefore, suppose that $\overline{\mathcal{P}}_{xy}|_{-\infty x}$ intersects $\mathcal{P}_{pq}$.

Now consider $\overline{\mathcal{P}}_{xz}|_{-\infty x}$. $\overline{\mathcal{P}}_{xz}|_{-\infty x}$ does not intersect $\mathcal{P}_{xz}|_{qz}$ since $\overline{\mathcal{P}}_{xz}$ is simple; if it intersects $\mathcal{P}_{yz}$, then we have as above that $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ is convex. Hence, we can also assume that $\overline{\mathcal{P}}_{xz}|_{-\infty x}$ intersects $\mathcal{P}_{pq}$ and as above we then have that $\mathcal{P}_{pq}|_{rq} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$.

On the other hand, if x is in $ext(\mathcal{P})$, then we consider the curve $\mathcal{P}' = \mathcal{P}_{pq}|_{rq} * \mathcal{P}_{xy}|_{qx} * \mathcal{P}_{xy}|_{xy_1} * \mathcal{P}_{yz}|_{y_1r}$. As in the discussion of Lemma 6.8 it can be seen that $int(\mathcal{P}')$ contains z_1. Hence, $\overline{\mathcal{P}}_{xz}$ intersects $\mathcal{P}'$ after z_1 in a point s. There are three cases to distinguish. If s belongs to $\mathcal{P}_{yz}|_{y_1r}$, then $q, r \in \mathcal{C}_d\text{-}hull(\{x,s\})$. Since $s \in \mathcal{C}_d\text{-}hull(\{y,z\})$, $\mathcal{P}_{pq}|_{rq}$ is contained in $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$. If s belongs to $\mathcal{P}_{xy}|_{xy_1}$, then $z \in \mathcal{C}_d\text{-}hull(\{x,y\})$ and $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$ is d-convex. If s belongs to $\mathcal{P}_{pq}$, then Lemma 6.9 yields that $\lambda_d(\mathcal{P}_{yz}|_{z_1r}) = \lambda_d(\overline{\mathcal{P}}_{xy}|_{z_1s}) + \lambda_d(\mathcal{P}_{pq}|_{sr})$. Hence, $s \in \mathcal{C}_d\text{-}hull(\{y,z\})$ and $\mathcal{P}_{pq}|_{rs} \subseteq \mathcal{C}_d\text{-}hull(\{y,z\})$. Further, $\mathcal{P}_{pq}|_{sq} \subseteq \mathcal{C}_d\text{-}hull(\{x,s\}) \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y,z\}))$.

This concludes the discussion of the Cases 2 and 3. Since a similar situtation will arise in Case 4 we state the result about $\mathcal{P}_1$ as a lemma.

Lemma 6.17 *Let $\mathcal{P}_{rq}$ be a d-straight path in $ext(\mathcal{C})$ with $r \in \mathcal{P}_{yz}$ and $q \in \mathcal{P}_{xz}$, then*

$\mathcal{P}_{rq} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$.

6.9.4 Case 4

We now assume that p is before y_1 on $\mathcal{P}_{xy}$ and q is before z_1 on $\mathcal{P}_{xz}$. We first suppose that $\mathcal{P}_{pq}$ is completely contained in $ext(\mathcal{C})$. We consider the curve $\mathcal{P} = \mathcal{P}_{xy}|_{xp} * \mathcal{P}_{pq} * \mathcal{P}_{xy}|_{qx}$. If z is in $ext(\mathcal{P})$, then by Lemma 6.16 we have that in this case $\mathcal{P}_{pq}$ belongs to $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$. In the same way as in the discussion of the Cases 2 and 3 we have by Lemma 6.7 and Lemma 6.15 that $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$ contains $\mathcal{P}_{pq}$ if z is in $int(\mathcal{P})$. This again leaves us with the only remaining possiblity that $\mathcal{P}_{pq}$ intersects $\mathcal{P}_{yz}$. Let r be the first intersection point of $\mathcal{P}_{pq}$ and $\mathcal{P}_{yz}$ on $\mathcal{P}_{pq}$ and s be the last. Clearly, $\mathcal{P}_{pq}|_{rs} \subseteq \mathcal{C}_d\text{-}hull(\{y, z\})$. Further, by Lemma 6.17 we have $\mathcal{P}_{pq}|_{sq} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$. The symmetry in p and $\mathcal{P}_{xy}$ with q and $\mathcal{P}_{xz}$ implies that also $\mathcal{P}_{pq}|_{pr} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$. Hence, $\mathcal{P}_{pq} = \mathcal{P}_{pq}|_{pr} * \mathcal{P}_{pq}|_{rs} * \mathcal{P}_{pq}|_{sq}$ is contained in $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$ and this concludes the discussion of the last case.

Above we have shown that, for any two points p and q in $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$ and any d-straight path $\mathcal{P}_{pq}$ from p to q, $\mathcal{P}_{pq} \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$. This implies that $\mathcal{C}_d\text{-}hull(\{p, q\}) \subseteq \mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$ and, by the completeness of $(\mathbf{X}, \mathcal{C}_d)$ we have that $\mathcal{C}_d\text{-}join(x, \mathcal{C}_d\text{-}hull(\{y, z\}))$ is d-convex which concludes the proof.

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