Symbolic Planning with Edge-Valued Multi-Valued Decision Diagrams - Detailed Proofs

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Abstract

This report contains the proof of correctness, soundness and optimality for EVMDD-A* presented in the paper *Symbolic Planning with Edge-Valued Multi-Valued Decision Diagrams* (Speck, Geißer, and Mattmüller 2018).

1 Transition Relation

Lemma 1. *Let (s, t') be an arbitrary state over $\mathcal{V} \cup \mathcal{V}'$. For any action a it holds that $(s, t') \in T_a$ iff a is applicable in s and $t = s[a]$.*

Proof. Let T'_a be the intermediate EVMDD of Terms (3) to (5). By construction of T'_a: a state $(s, t') \in T'_a$ iff a is applicable in s and $t = s[a]$. Furthermore, it holds that $(s, t') \in \mathcal{E}_{c_a}$ for all $(s, t') \in \mathcal{V} \cup \mathcal{V}'$ (Def. 1). Thus, $(s, t') \in T_a$ iff $(s, t') \in (T'_a \overset{\max}{\wedge} \mathcal{E}_{c_a})$ iff $(s, t') \in T'_a$ iff a is applicable in s and $t = s[a]$. □

Lemma 2. *Let $(s, t') \in T_a$. Then $T_a(s, t') = c_a(s)$.*

Proof. The intermediate EVMDD T'_a of Terms (3) to (5) contains only states with 0 or infinite cost (Def. 4 & Def. 5). Since $(s, t') \in T_a$, it holds that $T'_a(s, t') = 0$. Then, $T_a(s, t') = (T'_a \overset{\max}{\wedge} \mathcal{E}_{c_a})(s, t') = \max(T'_a(s, t'), c_a(s, t')) = \max(0, c_a(s, t')) = c_a(s, t') = c_a(s)$. □

2 Image

Note that we sometimes use "min" instead of $\overset{\min}{\vee}$. This simplifies the notations. If "min" is used for partial functions, we mean $\overset{\min}{\vee}$.

Theorem 1. *Let t be an arbitrary state over $\mathcal{V}$. Then $t \in \text{image}(\mathcal{E}, T_a)$ iff there exists a state $s \in \mathcal{E}$ such that a is applicable in s and $t = s[a]$.*

Proof.

$$
\begin{aligned}
& t \in \text{image}(\mathcal{E}, T_a) \\
& \Leftrightarrow t \in (\exists^{\text{LC}}_{\mathcal{V}}(\mathcal{E} + T_a))[\mathcal{V}' \leftrightarrow \mathcal{V}] && \text{(Definition 7)} \\
& \Leftrightarrow t' \in \exists^{\text{LC}}_{\mathcal{V}}(\mathcal{E} + T_a) && \text{(Substitution Lemma)} \\
& \Leftrightarrow t' \in \exists^{\text{LC}}_{v_1,\ldots,v_n}(\mathcal{E} + T_a) && \text{(Definition } \exists^{\text{LC}}) \\
& \Leftrightarrow \exists s : (s, t') \in (\mathcal{E} + T_a) && \text{(Transformation)} \\
& \Leftrightarrow \exists s : (s, t') \in \mathcal{E} \text{ and } (s, t') \in T_a && \text{(Definition 4)} \\
& \Leftrightarrow \exists s : s \in \mathcal{E} \text{ and } (s, t') \in T_a && \text{(Transformation)} \\
& \Leftrightarrow \exists s : s \in \mathcal{E} \text{ and } a \text{ is applicable in } s \text{ and } t = s[a] && \text{(Lemma 1)} \\
& \Leftrightarrow \text{there exists a state } s \in \mathcal{E} \text{ s.t. } a \text{ is applicable in } s && \text{(Transformation)} \\
& \qquad \text{and } t = s[a]
\end{aligned}
$$

□

From Theorem 1, Lemma 1 and Lemma 2 follows Corollary 1 which will be used to prove Theorem 2.

Corollary 1. *Let t be an arbitrary state over $\mathcal{V}$ with $t \in \text{image}(\mathcal{E}, T_a)$. Then there exists a state $s \in \mathcal{E}$ such that $(s, t') \in T_a$.*

Proof. By definition $t \in \text{image}(\mathcal{E}, T_a)$. Thus, by Theorem 1 there is a state $s \in \mathcal{E}$ such that a is applicable in s and $t = s[a]$. It follows that there exists a state $s \in \mathcal{E}$ such that $(s, t') \in T_a$ (Lemma 1). □

Theorem 2. *Let $\hat{\mathcal{E}} = \text{image}(\mathcal{E}, T_a)$. Then $\hat{\mathcal{E}}(t) = \min_s(\mathcal{E}(s) + c_a(s))$ for all states $t \in \hat{\mathcal{E}}$.*

Proof.

$$
\begin{aligned}
\hat{\mathcal{E}}(t) &= (\text{image}(\mathcal{E}, T_a))(t) \\
&= ((\exists^{\text{LC}}_{\mathcal{V}}(\mathcal{E} + T_a))[\mathcal{V}' \leftrightarrow \mathcal{V}])(t) && \text{(Definition 7)} \\
&= (\exists^{\text{LC}}_{\mathcal{V}}(\mathcal{E} + T_a))(t') && \text{(Substitution Lemma)} \\
&= (\exists^{\text{LC}}_{v_1,\ldots,v_n}(\mathcal{E} + T_a))(t') && \text{(Definition } \exists^{\text{LC}}) \\
&= (\min_{v_1,\ldots,v_n}(\mathcal{E} + T_a))(t') && \text{(Definition } \exists^{\text{LC}}) \\
&= (\min_{s}(\mathcal{E} + T_a))(t') && \text{(Transformation)} \\
&= (\min_{s}(\mathcal{E}(s, *) + T_a(s, *)))(t') && \text{(Transformation)} \\
&= \min_{s}(\mathcal{E}(s, t') + T_a(s, t')) && \text{(Transformation)} \\
&= \min_{s}(\mathcal{E}(s) + T_a(s, t')) && \text{(Transformation)} \\
&= \min_{s}(\mathcal{E}(s) + c_a(s)) && \text{(Corollary 1 + Lemma 2)}
\end{aligned}
$$

□

3 Preimage

Theorem 3. *Let s be an arbitrary state over $\mathcal{V}$. Then $s \in \text{preimage}(\hat{\mathcal{E}}, T_a)$ iff there exists a state $t \in \hat{\mathcal{E}}$ such that a is applicable in s and $t = s[a]$.*

Proof.

$$
\begin{aligned}
& s \in \text{preimage}(\hat{\mathcal{E}}, T_a) \\
& \Leftrightarrow s \in \exists^{\text{LC}}_{\mathcal{V}'}(\hat{\mathcal{E}}[\mathcal{V} \leftrightarrow \mathcal{V}'] + T_a) && \text{(Definition 7)} \\
& \Leftrightarrow s \in \exists^{\text{LC}}_{v'_1,\ldots,v'_n}(\hat{\mathcal{E}}[\mathcal{V} \leftrightarrow \mathcal{V}'] + T_a) && \text{(Definition } \exists^{\text{LC}}\text{)} \\
& \Leftrightarrow \exists t : (s, t') \in (\hat{\mathcal{E}}[\mathcal{V} \leftrightarrow \mathcal{V}'] + T_a) && \text{(Transformation)} \\
& \Leftrightarrow \exists t : (s, t') \in \hat{\mathcal{E}}[\mathcal{V} \leftrightarrow \mathcal{V}'] \text{ and } (s, t') \in T_a && \text{(Transformation)} \\
& \Leftrightarrow \exists t : (t, s') \in \hat{\mathcal{E}} \text{ and } (s, t') \in T_a && \text{(Substitution Lemma)} \\
& \Leftrightarrow \exists t : t \in \hat{\mathcal{E}} \text{ and } (s, t') \in T_a && \text{(Transformation)} \\
& \Leftrightarrow \exists t : t \in \hat{\mathcal{E}} \text{ and } a \text{ is applicable in } s \text{ and } t = s[a] && \text{(Lemma 1)} \\
& \Leftrightarrow \text{there exists a state } t \in \hat{\mathcal{E}} \text{ s.t. } a \text{ is applicable in } s && \text{(Transformation)} \\
& \quad \text{and } t = s[a]
\end{aligned}
$$

□

From Theorem 3, Lemma 1 and Lemma 2 follows Corollary 2 which will be used to prove Theorem 4.

Corollary 2. *Let s be an arbitrary state over $\mathcal{V}$ with $s \in \text{preimage}(\hat{\mathcal{E}}, T_a)$. Then there exists a state $t \in \hat{\mathcal{E}}$ such that $(s, t') \in T_a$.*

Proof. By definition $s \in \text{preimage}(\hat{\mathcal{E}}, T_a)$. Thus, by Theorem 3 there is a state $t \in \hat{\mathcal{E}}$ such that a is applicable in s and $t = s[a]$. It follows that there exists a state $t \in \hat{\mathcal{E}}$ such that $(s, t') \in T_a$ (Lemma 1). □

Theorem 4. *Let $\mathcal{E} = \text{preimage}(\hat{\mathcal{E}}, T_a)$. For any state $s \in \mathcal{E}$ it holds that $\mathcal{E}(s) = \hat{\mathcal{E}}(s[a]) + c_a(s)$.*

Proof.

$$
\begin{aligned}
\mathcal{E}(s) &= (\text{preimage}(\hat{\mathcal{E}}, T_a))(s) \\
&= (\exists^{\text{LC}}_{\mathcal{V}'}(\hat{\mathcal{E}}[\mathcal{V} \leftrightarrow \mathcal{V}'] + T_a))(s) && \text{(Definition 7)} \\
&= (\exists^{\text{LC}}_{v'_1,\dots,v'_n}(\hat{\mathcal{E}}[\mathcal{V} \leftrightarrow \mathcal{V}'] + T_a))(s) && \text{(Definition } \exists^{\text{LC}}) \\
&= (\min_{v'_1,\dots,v'_n}(\hat{\mathcal{E}}[\mathcal{V} \leftrightarrow \mathcal{V}'] + T_a))(s) && \text{(Definition } \exists^{\text{LC}}) \\
&= (\min_{t'}(\hat{\mathcal{E}}[\mathcal{V} \leftrightarrow \mathcal{V}'] + T_a))(s) && \text{(Transformation)} \\
&= (\min_{t}(\hat{\mathcal{E}}[\mathcal{V} \leftrightarrow \mathcal{V}'](*, t') + T_a(*, t')))(s) && \text{(Transformation)} \\
&= \min_{t}(\hat{\mathcal{E}}[\mathcal{V} \leftrightarrow \mathcal{V}'](s, t') + T_a(s, t')) && \text{(Transformation)} \\
&= \min_{t}(\hat{\mathcal{E}}(t, s') + T_a(s, t')) && \text{(Substitution Lemma)} \\
&= \min_{t}(\hat{\mathcal{E}}(t) + T_a(s, t')) && \text{(Transformation)} \\
&= \min_{s[a]}(\hat{\mathcal{E}}(s[a]) + c_a(s)) && \text{(Cor. 2 + Lem. 2 + Thm. 3)} \\
&= \hat{\mathcal{E}}(s[a]) + c_a(s) && \text{(Definition 1)}
\end{aligned}
$$

□

4 EVMDD-A*

Lemma 3. *Let Π be a planning task and h be a consistent heuristic. EVMDD-A* expands states in the same order and with the same g-values as A* with FIFO tie-breaking rule.*

Proof. Let S_f be all states with minimum f-value of an open list *Open*. Recall that in A* the tie-breaking between different states with minimum f-value in *Open* can be arbitrary. Let's assume the tie-breaking rule is "first in first out (FIFO)". The difference between EVMDD-A* and A* is that EVMDD-A* expands all states of S_f at once while A* iteratively ($|S_f|$ iterations) extracts these states. It is not possible that any other state is expanded before the $|S_f|$ iterations are finished, because h is consistent and therefore all newly generated successors have at least the f-value of all states in S_f.

- **Goal check.** Any ordering of expanding states in S_f is possible in A*. Thus, it is equivalent to first check if any state in S_f is a goal state.
- **Closed list.** Any ordering of expanding states in S_f is possible in A*. Thus, it is equivalent to first add all states S_f to the closed list and then expand all states S_f.
- **Open list.** By Theorem 1, in EVMDD-A*, all successors of S_f are generated and added to the open list if they are not contained in the closed list.

This is equivalent to adding them iteratively to *Open*. By Theorem 2 the cost of a successor $\hat{s}$ is the minimum cost with which $\hat{s}$ is reachable from any state in S_f applying action a. In line 9 (Algorithm 1), the minimum cost is taken from the current cost of $\hat{s}$ in *Open* or the minimum cost with which $\hat{s}$ is reachable from S_f applying any actions $a \in A$. Thus, the cost of a state $\hat{s}$ in *Open* is only updated iff it is reachable with lower cost from any expanded state in S_f. Again, this is equivalent to A* after $|S_f|$ iterations.

Therefore, EVMDD-A* and A* expand nodes in the same order and with the same g-values. □

Lemma 4. *Let* Π *be a planning task and* h *be a consistent heuristic. EVMDD-A* returns "no plan" iff A* returns "no plan".*

Proof. In EVMDD-A*, "no plan" is returned iff the open list is empty. By Lemma 3, the open list in EVMDD-A* is found empty iff the open list in A* is found empty. □

Lemma 5. *Let* Π *be a planning task and* h *be a consistent heuristic. If a plan exists for* Π*, EVMDD-A* returns the same plan as A* with FIFO tie-breaking rule.*

Proof. EVMDD-A* expands states in the same order and with the same g-values as A* (Lemma 3). Heuristic h is consistent, therefore all states in the closed list have minimum g-values g^*, i.e. the minimum cost with which they can be reached from s_0. ConstPlan is a version of backward greedy search with perfect heuristic $h^* = g^*$ where the g^*-values are stored in the closed list. Thus, ConstPlan and therefore EVMDD-A* returns an optimal plan from s_0 to any goal state expanded in EVMDD-A*. EVMDD-A* expands the same goal state as A* (Lemma 3). Thus, EVMDD-A* returns a plan iff A* returns a plan and EVMDD-A* returns the same plan as A* (if a plan exists). □

Theorem 5 & 6. *EVMDD-A* is complete, sound and optimal for consistent heuristics.*

Proof. Let Π be a planning task and h be a consistent heuristic. EVMDD-A* returns "no plan" iff A* returns "no plan" (Lemma 4). If a plan exists for Π, EVMDD-A* returns the same plan as A* (Lemma 5). EVMDD-A* is complete, sound and optimal for consistent heuristics because A* is it too. □

References

Speck, D.; Geißer, F.; and Mattmüller, R. 2018. Symbolic Planning with Edge-Valued Multi-Valued Decision Diagrams. In *Proceedings of the International Conference on Automated Planning and Scheduling (ICAPS)*. Accepted.