The Communication Cost of Information Spreading in Dynamic Networks*

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Abstract

This paper investigates the message complexity of distributed *information spreading* (a.k.a *gossip or token dissemination*) algorithms in adversarial dynamic networks. While distributed computations in dynamic networks have been studied intensively over the last many years, almost all of the existing work solely focuses on the time complexity of distributed algorithms. This paper focuses on the message complexity of information spreading in a dynamic network and presents several new results.

In information spreading, the goal is to spread k tokens of information to every node on an n-node network. We are interested in the *amortized* (average) message complexity of spreading a token, assuming that the number of tokens is large. In a static network, this basic problem can be solved using (asymptotically optimal) $O(n)$ amortized messages per token. Our focus is on token-forwarding algorithms, which do not manipulate tokens in any way other than storing, copying and forwarding them.

We assume that the dynamic sequence of network graphs is provided by an adaptive *worst-case* adversary that is aware of the status of all nodes and the algorithm (including the current random choices) and can rewire the network arbitrarily in every round with the constraint that it always keeps the n-node network connected. We present two sets of results depending on how nodes send messages to their neighbors:
1. *Local broadcast setting:* We show a tight lower bound of $\Omega(n^2)$ on the number of amortized local broadcasts, which is matched by the naive flooding algorithm.
2. *Unicast:* We study how the message complexity behaves as a function of the number of dynamic changes in the network. To facilitate this, we introduce a new and natural complexity measure for analyzing dynamic networks called *adversary-competitive message complexity* where the adversary pays a unit cost for every topological change. Under this model, it is shown that if k is sufficiently large, we can obtain an optimal amortized message complexity of $O(n)$. We also present a randomized algorithm that achieves *subquadratic* amortized message complexity when the number of tokens is not large under an *oblivious adversary*, which is same as the worst-case adversary, except that it is oblivious to the random choices made by the algorithm. Our analysis of the unicast communication under the adversary-competitive model (which is of independent interest) is a main contribution of this paper.

1 Introduction

Many modern distributed communication networks such as ad hoc wireless, sensor, and mobile networks, overlay and peer-to-peer (P2P) networks are inherently dynamic (suffer from a high rate of connections and disconnections) and bandwidth-constrained. Hence, understanding the possibilities and limitations of distributed computation in dynamic networks has been a major goal in recent years.

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In this paper, we study the fundamental problem of information spreading on (synchronous) dynamic networks. This problem was analyzed for static networks by Topkis [33], and was in particular studied on dynamic networks by Kuhn, Lynch, and Oshman [24]. In the information spreading problem (also called *k-gossip or k-token dissemination*), there are k pieces of information (tokens) that are initially present in some nodes and the problem is to disseminate the k tokens to all the n nodes in the network, under the bandwidth constraint that one token can go through an edge per round. This problem is a fundamental primitive for distributed computing; indeed, solving n-gossip, where each node starts with exactly one token, allows any function of the initial states of the nodes to be computed, assuming the nodes know n [24].

The dynamic network models that we consider in this paper allow a worst-case adversary known as *strongly adaptive* that can choose any communication links among the nodes for each round, with the only constraint being that the resulting communication graph be connected in each round; this adversary can choose the links with the knowledge of the tokens that any node can send in that round as well as its random choices (in one of the results we also consider an adversarial model that is oblivious to the random choices made by the algorithm). Our adversarial models are closely related to those adopted in recent studies (e.g., see [8, 16, 18, 24, 29, 14]). We distinguish two variants of the basic model, depending on whether nodes communicate by *local broadcast* (i.e., a node always sends the same message to all its neighbors) or whether we allow nodes to do *unicast communication* (i.e., nodes can possibly send different messages to different neighbors in the same round). For more information on the model, we refer to Section 1.3. We note that most of the prior work (e.g., [18, 24, 29]) only considered communication by local broadcast.

The focus of the present paper is on *token-forwarding* algorithms, which do not manipulate tokens in any way other than storing, copying, and forwarding them. Token-forwarding algorithms are simple and easy to implement and have been widely studied (e.g, see [27, 30]). The paper investigates the *message complexity* of token-forwarding algorithms for information spreading. Message complexity—the total number of messages sent by all nodes during the course of an algorithm—is an important performance measure. It directly relates to the cost of communication, which is a dominant cost in many real-world settings (e.g., it is correlated to energy, power, etc. in wireless networks). While information spreading in dynamic networks have been studied intensively over the last years, almost all of the existing work (e.g., [5, 8, 16, 18, 22, 24]) solely focuses on the *time (round) complexity* of distributed algorithms. In many cases, the currently best algorithms for information spreading in adversarial dynamic networks have a high message complexity and in many cases, a high time complexity as well. In contrast, in this paper, we are interested in the *amortized* message complexity of information spreading, i.e., the average message cost of spreading k tokens (when k is large) in a dynamic network. To the best of our knowledge, this aspect has not been studied in prior works on information spreading in dynamic networks (cf. Section 1.2).

In any n-node static network, a simple token-forwarding algorithm that pipelines token transmissions up a rooted spanning tree, and then broadcasts them down the tree completes k-gossip in $O(n + k)$ rounds [30], which is clearly asymptotically tight because the diameter of the network might be $\Theta(n)$ and because every node to receive k different tokens. In fact, $O(n + k)$ rounds are even sufficient if in each round, each node forwards an arbitrary not yet forwarded token to each of its neighbors [33]. In a dynamic network, it is known that under a strongly adaptive adversary and if the communication is via local broadcast, the $O(n + k)$ bound cannot be achieved; Dutta et al. [18] (see also [22]) showed that $\Omega(nk/\log(nk)+n)$ rounds are necessary. This bound is essentially tight (up to a logarithmic factor), since one can easily achieve an upper bound of $O(nk)$ by flooding. We do not know any tight bounds on the time complexity for *unicast* communication.

With regard to messages, we are interested in the *amortized* (average) message complexity of spreading a token. In a static network, one can first build a spanning tree (which can take as much as $\Omega(n^2)$ messages[1] in

[1]This bound is true in the KT0 model where nodes don't have initial knowledge of their neighbors' IDs. On the other hand, in the KT1 model, where each node has initial knowledge of the IDs of their respective neighbors, it is possible to build a spanning tree in $O(n \text{ polylog}(n))$ messages [23]. Note that this distinction is not very important in the amortized setting in a static network, since in both cases the amortized message complexity is $O(n)$ if $k = \Omega(n)$. In the dynamic setting, we essentially assume the KT1 model for unicast communication, whereas for broadcast communication the distinction is not important, see Section 1.3 for more details.

graphs with $\Theta(n^2)$ edges[26]), and then using the spanning tree edges to disseminate the tokens to all nodes; this takes $O(n^2+nk)$ messages overall or $O(n^2/k+n)$ amortized messages per token. If k is sufficiently large[2], say at least n, then the above bound gives $O(n)$ amortized messages per token, which is optimal (since each node has to receive the token). On the other hand, for dynamic networks, the situation is far less clear. In the case of *local broadcast* communication (where each broadcast is counted as one message[3]), a $\Theta(n^2)$ amortized message upper bound per token is straightforward to obtain by using flooding (each node broadcasts each token for n rounds). For unicast communication (cf. Section 1.3), again an $O(n^2)$ amortized upper bound is easy to obtain (each node sends each token at most once to each other node; note that for unicast communication each message to a neighbor is counted as one message). In both cases, non-trivial lower bounds are not known. Thus, the *central question that we seek to address in this work is whether one can achieve $o(n^2)$ or even asymptotically optimal $O(n)$ amortized message complexity when k is large (for both the local broadcast and the unicast settings)*. We note that prior works (including [5, 8, 16, 18, 22, 24]) do not address this question.

1.1 Our Main Results

In the local broadcast setting, we give a negative answer to the above question and show that with a strongly adaptive adversary, the $\Theta(n^2)$ amortized message complexity bound of the naive algorithm is indeed necessary (cf. Section 2). This "bad" bound for local broadcast is a motivation for considering the unicast setting. For the unicast setting, we study how the message complexity behaves as a *function of the number of dynamic changes in the network*. To facilitate this, we introduce a new and natural complexity measure for analyzing dynamic networks called *adversary-competitive message complexity* (cf. Definition 1.3). While the adversary is free to change the topology arbitrarily from round to round, this measure allows one to intuitively assume that it has to pay some price for every connection and reconnection and we allow an algorithm a "free" communication budget of comparable size.[4] Under this new complexity measure (defined formally in Section 1.3), we show that if k is sufficiently large, we obtain an optimal amortized message complexity of $O(n)$ (cf. Section 3). In case the dynamic network topology satisfies some natural additional properties, we also show that the algorithm terminates in $O(nk)$ rounds. We present two algorithms in this setting depending on how the tokens are initially distributed: (1) a *single-source case*, where all the tokens start at the same node and (2) a *multi-source case*, where the initial token distribution is arbitrary.

When the number of tokens is not very large, say $k = n$ (i.e., n-gossip), the $O(n)$ amortized bound does not hold. In this setting, we are able to show a *subquadratic* amortized message complexity under an *oblivious adversary*, which is same as the worst-case adversary, except that it is oblivious to the random choices made by the algorithm (cf. Section 3.2.2). Our algorithm is randomized and is based on random walks.

Our analysis of the unicast communication under the adversary-competitive model is a main contribution of this paper. We believe that the adversary-competitive model can be useful alternative to the current models in analyzing various other important problems such as leader election and agreement in dynamic networks (see e.g., [6, 7]).

Our work raises several key open questions that are discussed in Section 4.

1.2 Related work and Comparison

Information spreading (or dissemination) in networks is a fundamental problem in distributed computing and has a rich literature. The problem is generally well-understood on static networks, both for interconnection networks [27] as well as general networks [4, 28, 30]. In particular, the k-gossip problem can be solved in $O(n + k)$ rounds on any n-node static network [33]. There also have been several papers on broadcasting, multicasting, and related problems in static heterogeneous and wireless networks (e.g., see [3, 12, 13, 15]).

[2]There are natural applications where k is large, e.g., if all nodes have some token to broadcast or if some node has a stream of messages as for example in video transmissions.

[3]This is reasonable, especially, in the context of wireless networks where nodes communicate by local broadcast.

[4]For example, due to the actions of the lower layer link protocol (that is responsible to establish the connection when a physical link comes up), one can assume that whenever a new edge is created, some information is exchanged anyhow by the link layer.

Dynamic networks have been studied extensively over the past three decades. Early studies focused on dynamics that arise when edges or nodes fail. A number of fault models, varying according to extent and nature (e.g., probabilistic vs. worst-case) of faults allowed, and the resulting dynamic networks have been analyzed (e.g., see [4, 28]). There have been several studies that constrain the rate at which changes occur, or assume that the network eventually stabilizes (e.g., see [1, 17, 19]).

To address highly unpredictable network dynamics, stronger adversarial models have been studied by [8, 24, 29] and others; see the recent survey of [14] and the references therein. Unlike prior models on dynamic networks, these models and ours do not assume that the network eventually stops changing; the algorithms are required to work correctly and terminate even in networks that change continually over time.

The model of [18, 22, 24] allows for a much stronger adversary than the ones considered in past work [9, 10, 11]. In particular, the work of [18] (also see [22]), showed that every token forwarding information spreading algorithm that uses local broadcast for communication under a strongly adaptive adversary (the same as considered in this paper — cf. Section 2) requires $\Omega(n^2/\log n)$ rounds to complete. The survey of [25] summarizes recent work on dynamic networks.

Recent work of [20, 21] presents information spreading algorithms based on network coding [2]. As mentioned earlier, one of their important results is that the k-gossip problem on the adversarial model of [24] can be solved using network coding in $O(n+k)$ rounds assuming the token sizes are sufficiently large ($\Omega(n \log n)$ bits).

It is important to note that *all the above results* deal with the *time complexity* of information spreading in dynamic networks and not the message complexity. In particular, the works of [18, 16] (as well as other works discussed above) the focus is on the time complexity (i.e., the number of rounds needed for spreading k tokens in a n-node dynamic network), whereas the focus here is on *amortized* message complexity for spreading k tokens. We note that there is an important difference between the two measures. In particular, algorithms with efficient time complexity need not necessarily be message-efficient and vice-versa and hence prior time complexity-based results do not directly imply the results of this paper. Indeed, one can exchange up to $\Theta(n^2)$ messages (in a graph with $\Theta(n^2)$ edges) in just one round, and since one needs at least $\Omega(n)$ rounds for information spreading (in the worst-case), the total message complexity can be as high as $\Omega(n^3)$ (for unicast). In other words, a message-efficient algorithm can take a longer time but exchanging less total number of messages, e.g., by sending messages only along a few edges and/or by using silence. However, as we show in Section 2, the amortized message complexity lower bound (even) for local broadcast (where a node's local broadcast to all its neighbors is counted as just one message) is close to the worst possible, i.e., $\Omega(n^2/\log n)$. The proof for this lower bound is inspired by the time complexity lower bound of [18], although the two proofs differ in their details. The "bad" lower bound for local broadcast motivates considering unicast communication which is the main focus of this paper. It is important to point out another difference between amortized time complexity and amortized message complexity. While amortized time complexity can be as low as $\Omega(D)$ (where D is the network diameter, which can be much smaller than n), the amortized message complexity is at least (trivially) $\Omega(n)$, since a token has to reach all the n nodes. There has not been much progress on improving time complexity (total or amortized) in dynamic networks (both for unicast and local broadcast) in the oblivious adversary model in general networks, although prior works [16, 5] has achieved improved (subquadratic in n) total *time* complexity under additional assumptions on the dynamic network model (these are different from what is considered here). In particular, the work of [16] considers a dynamic network and presents an information spreading algorithm that can have subquadratic *time* complexity under some restricted conditions, e.g., when the dynamic mixing time (defined in [16]) is small. The work of [16] does not address amortized message complexity at all and the result in the oblivious adversary setting of this paper do not follow from the results of [16]. Both papers use techniques based on random walks (which are very useful in the oblivious setting) which were originally developed in [31, 32], but the algorithms are quite different.

While the work of [18] adopts the strongly adversarial model and local broadcast communication (we adopt the same model here for the local broadcast communication —cf. Section 2), the work of [16, 5] adopts the oblivious adversary model (we also adopt the oblivious model here for unicast communication in Section 3.2.2), a *novel aspect of this paper is introducing and adopting a new communication cost model* that measures the communication cost of an algorithm as a function of the amount of topological changes that occur in a given execution and a new message complexity measure called *adversary-competitive message complexity* (Section 1.3). A main contribution of this paper is showing that under this new complexity measure, one can obtain an efficient amortized message complexity for *unicast* communication that is significantly better than the worst-case bound of $\Omega(n^2)$.

1.3 Dynamic Network, Communication, and Cost Model

In the following, we formally define the dynamic network model, the communication models we consider, as well as the way in which we measure the communication cost (or message complexity) of a given token dissemination algorithm.

Dynamic Network Model: We model the network as a synchronous dynamic graph G with a fixed set of nodes V. Nodes communicate in synchronous rounds where round r starts at time $r-1$ and ends at time r. For any integer $r \geq 1$, we use $G_r = (V, E_r)$ to denote the graph of round r. Throughout we use $n := |V|$ to denote the number of nodes and $m_r := |E_r|$ to denote the number of edges in round r. For convenience, we define $E_0 := \emptyset$ and thus G_0 is the empty graph $(V, \emptyset)$. For every $r \geq 0$, we call $E_r^+ := E_r \setminus E_{r-1}$ the set of edges *inserted in round* r and we call $E_r^- := E_{r-1} \setminus E_r$ the set of edges *removed in round* r.

In order to always allow progress when globally broadcasting a message, we assume that each graph G_r is connected for $r \geq 1$. We sometimes also need the property that every edge which gets inserted remains in the graph for at least a given number of rounds. For an integer $\sigma \geq 1$, we call a graph *σ-edge stable* if for every $r \geq 1$ and every edge $e \in E_r$, there exists a round $r' \geq \max\{1, r-\sigma+1\}$ such that $e \in E_{r'} \cap \cdots \cap E_{r'+\sigma-1}$. Hence, after it appears, every edge remains in the graph for at least σ consecutive rounds. Note that every dynamic graph is 1-edge stable.

We assume that the dynamic topology is provided by a worst-case adversary. There are adversaries of different strengths, depending on the capability of adaptively reacting to random choices of a given algorithm. In this paper, we distinguish between a *strongly adaptive adversary* and an *oblivous adversary*. The strongly adaptive adversary knows the algorithm's randomness of the current round in order to determine the dynamic topology for that round.[5]The oblivious adversary is oblivous to any randomness used by the algorithm, i.e., it has to commit to the sequence of network topologies before the execution of a distributed algorithm starts. Note that for deterministic algorithms, both adversaries are the same.

Communication Model: Throughout the paper, we assume that each node $v \in V$ has a unique $O(\log n)$-bit identifier $\mathsf{ID}(v)$ and that in each round, each node can send messages containing a constant number of tokens and $O(\log n)$ additional bits to its neighbors. We distinguish different modes of communication, depending on whether the message exchange among neighbors is based on local broadcast or on unicast.

(I) **Local Broadcast Communication:**

In each round r, each node v can locally broadcast a message which is received by all neighbors of v. Node v learns the set of neighbors in round r when receiving the round r messages from them.

(II) **Unicast Communication:**

At the beginning of each round r, each node v is informed about the IDs of its neigbors in round r. Node v can then send a different message to each neighbor.

Note that if the neighborhood information is not available instantaneously, it can be obtained by exchanging messages. As a consequence, in a 2-edge stable dynamic graph, the known neighborhood information and unknown neighborhood information are equivalent with a cost of extra messages.

[5]In comparison, a *weakly adaptive adversary* only knows the algorithm's randomness up to the round before the current round.

Communication Cost: The *communication cost* of a protocol is measured by its *message complexity*, i.e., by the total number of messages sent by all the nodes throughout the whole execution. Formally, for the above communication models, the message complexity of a distributed algorithm is defined as follows.

Definition 1.1 (Message Complexity). *The* message complexity *of a distributed algorithm is the* total number of messages *sent in a worst-case execution. If communication is by local broadcast, each local broadcast by some node counts as one message. If communication is by unicast, messages to different neighbors are counted separately.*

Let us now define the problem formally. The main focus of this article is to study the message complexity of solving this problem.

Definition 1.2 (k-Token Dissemination Problem). *For some positive integer k, k distinct tokens are initially placed at some nodes in the network. The goal is to disseminate all the k tokens to all the nodes in the network.*

As discussed in Section 1, we are particularly interested in how the extent of dynamic topology changes affects the communication cost of token dissemination. We therefore consider a cost model that measures the communication cost of an algorithm as a function of the amount of topological changes that occur in a given execution. We formally define the *number of topological changes* $\mathsf{TC}(\mathcal{E})$ of an execution $\mathcal{E}$ as the total number of edges that are inserted throughout an execution, i.e., for an x-round execution $\mathcal{E}$ with dynamic graph $G_r = (V, E_r)$, we have $\mathsf{TC}(\mathcal{E}) := \sum_{r=1}^{x} |E_r^+|$.[6] The following definition captures the notion that for each dynamic change caused by the dynamic network adversary, a distributed algorithm is allowed to send a given number of messages "for free".

Definition 1.3 (Adversary-Competitive Message Complexity). *Given a parameter $\alpha \geq 0$, we say that a distributed algorithm has α-*adversary-competitive message complexity *M if for every execution $\mathcal{E}$, the total message complexity of the algorithm is upper bounded by $M + \alpha \cdot \mathsf{TC}(\mathcal{E})$.*

To capture the progress of an algorithm, one way is to count how many new tokens have been received so far by the nodes.

Definition 1.4 (Token Learning). *A token learning is an event denoted by a tuple $\langle v, \tau, r\rangle$. Event $\langle v, \tau, r\rangle$ occurs in some x-round execution $\mathcal{E}$, if and only if node v receives token τ for the first time in round r, where $r \leq x$. Then, we say v learns τ in round r.*

Based on the above definition, if each of the k tokens is initially given to exactly one of the n nodes, it is trivial that $k(n-1)$ token learnings must occur during an algorithm execution solving k-token dissemination.

General Notation: Throughout the paper, we use $\log x := \log_2 x$ to denote logarithms to base 2.

2 Local Broadcast Model

In this section, we present a lower bound for the amortized message complexity of disseminating k tokens in the local broadcast communication model. We assume that each of the k tokens can initially be given to an arbitrary subset of the nodes with the only restriction that the nodes initially have at most $k/2$ tokens on average. We further assume that k is at most polynomially large in n. Our lower bound is an extension of the time complexity lower bound, which was developed by Dutta et al. in [18] and which was slightly generalized and simplified in [22]. The main idea of the lower bound is as follows. If initially, each token is given to each node independently with a constant probability, the lower bound shows that in each round of any k-token dissemination algorithm execution, a strongly adaptive adversary can enforce that in total at most $O(\log n)$ tokens are learned by the nodes. Because by the end of an execution, the nodes together need to learn $\Theta(nk)$

[6]Note that since we assume that at time 0 we start with an empty graph, the total number of edge deletions is always upper bounded by the total number of edge insertions. Hence we only count the edge insertions and not the edge deletions.

tokens (each node needs to learn the tokens it does not know initially), this directly implies a $\Omega(nk/\log n)$ time complexity lower bound. Here, we adapt the technique of the lower bound of [18, 22] to show that in any round with at most $O(n/\log n)$ broadcasting nodes[7], a strongly adaptive adversary can prevent any new tokens from being learned. Because the nodes together need to learn $\Theta(nk)$ tokens, together with the upper bound of $O(\log n)$ on the number of tokens learned in a single round, this implies that a strongly adaptive adversary can force any token dissemination algorithm to require at least $\Omega(nk/\log n)$ rounds with at least $\Omega(n/\log n)$ broadcasting nodes. This leads to the overall message complexity of at least $\Omega(n^2k/\log^2 n)$.

To prove our lower bound, we mostly use the notation in [22]. Let $\mathcal{T}$ denote the set of k tokens, and for each node $v \in V$, let $K_v(t)$ be the set of tokens that node v knows by time t. In each round r, let $i_v(r)$ denote the token broadcast by node v if v is a broadcasting node in round r. If v is not a broadcasting node in round r, we define $i_v(r) := \perp$. Note that a strongly adaptive adversary can determine the dynamic graph topology of round r after each node has chosen the token $i_v(r)$ to locally broadcast. Generally, a collection of pairs (v, i_v), where $v \in V$ and $i_v \in \mathcal{T} \cup \{\perp\}$ is called a token assignment.

In addition, the adversary determines a token set $K'_v \subseteq \mathcal{T}$ for each node v. The sets K'_v are just used for the analysis. Informally, one can think of K'_v as an additional set of tokens that node v knows at time 0. Formally, we do not assume that node v knows the tokens in K'_v initially, but whenever v learns a token from K'_v, we do not count this as progress (i.e., for node v, we only count how many tokens from $\mathcal{T} \setminus K'_v$ it has learned). To formally measure the progress, we define a potential function $\Phi(t) := \sum_{v\in V} |K_v(t) \cup K'_v|$. Recall that we assume that initially on average, each node knows at most $k/2$ tokens, i.e., $\sum_{v\in V} |K_v(0)| \leq nk/2$. The adversary chooses the sets K'_v in such a way that $\Phi(0) \leq 0.8nk$. In order to solve the token dissemination problem, the potential has to grow to nk. The choice of the sets K'_v therefore guarantees that the potential needs to grow by at least $0.2nk$ throughout the execution of a k-token dissemination protocol.

To study the growth of the potential function, the following notion is used. An (potential) edge $\{u, v\}$ is called *free* in round r, if and only if the communication over $\{u, v\}$ does not contribute to $\Phi(r) - \Phi(r-1)$, i.e., $\{u, v\}$ is free if and only if $i_u(r) \in \{\perp\} \cup K_v(r-1) \cup K'_v$ and $i_v(r) \in \{\perp\} \cup K_u(r-1) \cup K'_u$. Otherwise, the edge $\{u, v\}$ is called *non-free*. When determining the topology of round r, a strongly adaptive adversary can always add all free edges to the graph G_r without causing any increase of the potential function. If after adding all free edges, the graph has ℓ connected components, the adversary needs to add $\ell - 1$ additional edges "non-free" edges in order to make G_r connected. The potential function can then grow by at most $2(\ell - 1)$ because over each of these additional $\ell - 1$ edges, one token can be learned in each direction. In [18, 22], it is shown using a probabilistic method that the sets K'_v can be chosen such that $\Phi(0) \leq 0.8nk$ and such that in each round, the graph induced by only the free edges has at most $O(\log n)$ connected components. Every algorithm therefore needs at least $\Omega(nk/\log n)$ rounds for the potential to grow to nk.

The following lemma from [22] shows that if each token is randomly added to each set K'_v independently with probability $1/4$, adding all free edges reduces the number of components to $O(\log n)$ for all rounds with constant probability.

Lemma 2.1. *(Lemma 1 of [22]) If each set K'_u contains each token $i \in \mathcal{T}$ independently with probability $1/4$, with probability at least $3/4$, for all rounds r and all possible token assignments $(v, i_v(r))$ in round r, the graph $F(r)$ induced by all free edges in round r has at most $O(\log n)$ connected components.*

We next show that if the number of broadcasting nodes is small, adding all free edges leaves only one connected component. For a constant $c > 0$, we define a token assignment (v, i_v) to be c-sparse if at most $n/(c \log n)$ of the nodes are broadcasting nodes (i.e., for at most $n/(c \log n)$ nodes, we have $i_v \neq \perp$).

Lemma 2.2. *There is a constant $c > 0$ such that if each set K'_u contains each token $i \in \mathcal{T}$ independently with probability $1/4$, with probability at least $1 - 2^{-n}$, for all rounds r and all possible c-sparse token assignments $(v, i_v(r))$, the graph $F(r)$ induced by all free edges in round r consists of a single connected component.*

[7] Throughout this section, we call a node that performs a local broadcast in some round r, a broadcasting node in round r.

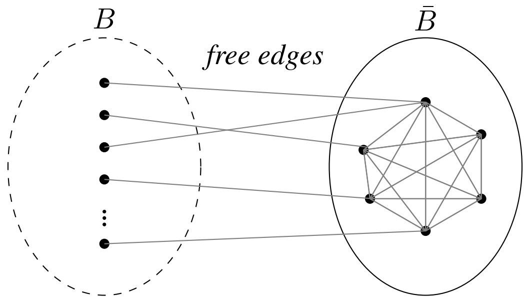

Figure 1: It shows the connected graph induced by (a subset of) the free edges in a round with at most $O(n/\log n)$ broadcasting nodes. The free edges among the nodes in $\bar{B}$ induce a clique, and each of the broadcasting nodes in B is connected to some node in $\bar{B}$ by a free edge.

Proof. We first bound the probability for a fixed c-sparse token assignment (v, i_v). The claim of the lemma will then follow by a union bound over all the possible c-sparse token assignments. Let B denote the set of broadcasting nodes, i.e., the nodes for which $i_v \neq \perp$. Further, let $\beta := |B| \leq n/(c \log n)$ and let $\bar{B} := V \setminus B$. Clearly, all the edges among the nodes in $\bar{B}$ are free. It is therefore sufficient to show that for each node v in B, there is a free edge connecting v to a node in $\bar{B}$. Then, all the free edges induce a connected graph over all the nodes (also see Figure 1).

Consider an edge $\{u, v\}$, where $u \in \bar{B}$, $v \in B$, and v is locally broadcasting token τ. Edge $\{u, v\}$ is a free round (for every round r) if $\tau \in K'_u(0)$. This happens with probability $1/4$ (independently for every node $u \in \bar{B}$). The probability that v has no free edge to some node in $\bar{B}$ is thus at most $(1/4)^{n-\beta}$. Thus, the probability that there exists at least one node in B that has no free edge to $\bar{B}$ is at most $\beta/4^{n-\beta}$. Considering a union bound over all $\binom{n}{\beta} < n^\beta$ ways to choose a set of β nodes and all at most k^β ways to choose the tokens to be sent out by these nodes, the probability that there exists a token assignment for which there is a node in B that has no free edge to $\bar{B}$ can therefore be upper bounded by

$$\begin{aligned}
&\Pr(\exists v \in B \textit{ s.t. } \forall u \in \bar{B} : \{v, u\} \textit{ is non-free}) \\
&\qquad \leq n^\beta \cdot k^\beta \cdot \frac{\beta}{4^{n-\beta}} \\
&\qquad = 2^{\beta \log(nk) + \log \beta - 2(n-\beta)} \\
&\qquad = 2^{\beta(\log(nk)+2) + \log \beta - 2n} \\
&\qquad \leq 4^{\frac{c}{2}\beta \log n - n} && \text{[for some constant } c \text{ and } k \text{ polynomial in } n\text{]} \\
&\qquad < 2^{-n} && \text{[for } \beta < \frac{n}{c \log n}\text{]}
\end{aligned}$$

Hence, with probability at least $1 - 2^{-n}$, for each possible token assignment (and for each round), each node $v \in B$ has a free edge connected to some node in $\bar{B}$. $\square$

Theorem 2.3. *In any always connected dynamic network, if initially each node an average knows at most half of the k tokens, the amortized message complexity of solving the k-token dissemination problem against a strongly adaptive adversary is at least $\Omega(n^2/\log^2 n)$ in the local broadcast communication model.*

Proof. Using the probabilistic method, we show that the adversary can choose the sets K'_u such that at time 0, $\Phi(0) \leq 0.8nk$ and such that for every possible strategy of the algorithm, the adversary can choose the graph of each round such that (1) the graph is connected, (2) the number of connected components after adding all free edges is at most $O(\log n)$, and (3) if there are at most $n/(c \log n)$ broadcasting nodes, for a sufficiently large constant $c > 0$, the free edges induce a connected graph. The theorem then follows because (a) the potential needs to grow by $0.2nk$ in order to solve the token dissemination problem and (b) the potential increase per round is always at most $O(\log n)$ and it is 0 if the number of broadcasting nodes is less than $n/(c \log n)$.

To apply the probabilistic method, we let each set K'_u contain each token $i \in \mathcal{T}$ independently with probability $1/4$. First note that by a standard Chernoff argument, the probability that $\sum |K'_u| > 0.3nk$ is exponentially small in nk and thus the probability that $\Phi(0) > 0.8nk$ is also exponentially small in nk. Further, by Lemma 2.1 and Lemma 2.2, for every round r, and every token assignment $(v, i_v(r))$, the graph $F(r)$ induced by all the free edges has the following two properties with probability at least $\frac{3}{4} - 2^{-n}$: (1) $F(r)$ contains at most $O(\log n)$ connected components, (2) $F(r)$ is connected over all the nodes if there are at most $n/(c \log n)$ broadcasting nodes. This shows that (for sufficiently large n), there is a way to choose the sets K'_u sets such that $\Phi(0) \leq 0.8nk$, the potential increase per round is at most $O(\log n)$, and if there are at most $n/(c \log n)$ broadcasting nodes, the potential increase is 0 and the claim of the theorem follows. □

3 Unicast Model

3.1 Single Source Node

Consider the k-token dissemination problem such that all the k tokens are initially given to a single source node. Let us now present a deterministic algorithm to solve this problem with message complexity of $O(n^2 + nk) + \mathsf{TC}(\mathcal{E})$ against a strongly adaptive adversary. Hence, the algorithm has *1-adversary-competitive message complexity* of $O(n^2 + nk)$ (cf. Def. 1.3). In other words, if the algorithm is provided with a budget that equals to the number of topological changes, then for sufficiently large k, the amortized message complexity to disseminate the tokens is linear in n. Note that even in a static graph, the cost to disseminate a single token is $\Omega(n)$. Hence, if the number of tokens is at least linear in n, the amortized message complexity is asymptotically best possible. Before we present the algorithm and its analysis, consider the following definitions.

Definition 3.1 (Complete and Incomplete Node)**.** *We say that node v is complete at time t if it has all the k tokens at this time. Otherwise, v is incomplete.*

Definition 3.2 (Bridge Node)**.** *In each round, any incomplete node that has a complete neighbor is called a bridge node for that round.*

3.1.1 Single-Source Unicast Algorithm

The source node considers an arbitrary order of the tokens and assigns integer i to its i^{th} token as its *token ID*. The algorithm lets only the complete nodes send the tokens during the execution. To this end, each complete node announces its completeness to its neighbors during the execution. In each round, each incomplete node sends token requests to (some of) its complete neighbors. Then, in the very next round, each complete node sends back the requested tokens to the requesting nodes if it is still connected to them. Although the general idea is simple, there is a need of a careful strategy to avoid redundant communication.

Each complete node v informs each node about the time of v's completeness at most once by remembering which nodes v informed before. Each node also remembers all the complete nodes it is informed by about their completeness. Each incomplete node chooses among its complete neihbors for sending token requests to, based on a priority defined by the following categorization of its adjacent edges.

Consider any edge $e = \{v, w\} \in E_r$, such that v is incomplete and w is complete. Then, e is called *new* in round r if the edge is inserted at the beginning of round r or $r - 1$. Edge e is called *contributive* if it is not new, but a new token is sent over it between the last insertion of the edge and the end of round r, i.e., it contributes to the dissemination. Otherwise, if e is neither new nor contributive, it is called *idle* in round r.

Based on the above definitions, if v has τ missing tokens, it creates τ token requests, one for each missing token. Then, v assigns one and only one distinct token request to each of the new edges (if any). Afterwards, if there are still token requests left to be assigned, v assigns exactly one request to each of the idle edges (if any). Finally, v does the same for the contributive edges. Note that since each edge has at most one assigned token request, there might be token requests that are not assigned in the current round. At the end, v sends the assigned token requests in round r over the corresponding edges.

Note that for categorizing an adjacent edge $e = \{v, w\}$, an incomplete node v might need to know whether it learns a token over e in round r or not. However, if v sends a token request over e in round $r-1$, and $e \in E_r$, then v knows that it learns a token over e in round r. Moreover, to avoid sending redundant token requests, node v needs to know whether it learns some requested token in round r or not. However, v knows the token requests it sent over its adjacent edges in round $r-1$. Then, by knowing the adjacent edges in round r, and the fact that complete nodes immediately respond to requests, v knows what tokens it learns in round r.

3.1.2 Analysis

First, let us argue the message complexity of the algorithm. Then, we show that with a natural stability assumption the time complexity is also small.

Theorem 3.1. *Given k tokens to disseminate in a dynamic network against a strongly adaptive adversary, the Single-Source Unicast Algorithm has 1-adversary-competitive message complexity of $O(n^2 + nk)$.*

Proof. There are three different types of messages sent by nodes during the algorithm execution; (1) token, (2) completeness announcement, and (3) token request. Each node sends the request of each distinct token to at most one neighbor in a round. If the connection to that complete neighbor remains for the very next round, then the requested token will be successfully received by the node and the node stops sending this token request. Therefore, each distinct token is received by each node once, and hence there are at most $O(nk)$ sent messages of type 1 throughout the execution.

Each of the n nodes informs at most $n-1$ other nodes about its completeness throughout the execution. Since each node avoids informing the same node more than once, at most $O(n^2)$ messages of type 2 are sent throughout the execution.

It remains to show that the number of sent messages of type 3 is at most $O(nk) + \mathsf{TC}(\mathcal{E})$ during execution $\mathcal{E}$. In each round where a token request is sent by some node, a new token is received in the next round unless the edge is removed. Therefore, we can say that the number of token requests sent at any time is at most $O(nk)$ plus the number of edge deletions. $O(nk)$ comes from the fact that there exist k tokens and each token is received by at most $O(n)$ nodes, each token once. Furthermore, since we assume that the initial graph is an empty graph, the number of edge deletion is upper bounded by $\mathsf{TC}(\mathcal{E})$. $\square$

In the following, we argue that with a natural stability assumption, the algorithm disseminates all the tokens and terminates fast. The following two lemmas show that prioritization of sending token requests over different edge types ensures fast dissemination.

Definition 3.3 (Futile Round)**.** *Round r is a futile round, if no token request is sent over a contributive edge in round r, and no token learning occurs in rounds $r+1$ and $r+2$.*

Lemma 3.2. *Let r be an arbitrary futile round in any execution of the Single-Source Unicast Algorithm on a 3-edge stable dynamic network. Then, if there exist ℓ bridge nodes in round r, at least ℓ idle edges are removed at the end of round r.*

Proof. First, let us show that every bridge node has an adjacent idle edge in round r. If there exists a new edge in round r, due to the 3-edge stability property and the higher priority of sending requests on new edges, a token is learned in at least one of rounds $r+1$ or $r+2$. Hence, there exists no new edge in round r. Now, for the sake of contradiction, let us assume that there exists a bridge node b in round r that does not have an adjacent idle edge. Since b cannot have an adjacent new edge either, it must have at least one contributive edge. Therefore, b sends a request over at least one of its contributive edges in round r, contradicting the assumption that r is a futile round.

Since every bridge node has an idle edge and no new edge, due to the mentioned priority rules, a bridge node sends a request over at least one of its idle edges. Since no new token is learned in round $r+1$, the idle edge carrying a request must be removed. Hence, from each bridge node at least one idle edge is removed at the end of round r. □

Lemma 3.3. *In any execution of the Single-Source Unicast Algorithm on a 3-edge stable n-node dynamic network, there exist at most n futile rounds until the last token request is sent.*

Proof. Let us first argue that it is not possible for a new edge to become idle. For any round $r > 0$, consider an arbitrary new edge $e = \{u, v\} \in E_r^+$, where u is complete and v is incomplete. Then in round $r+2$, either e is contributive or v is complete. Because, the only case that v does not send a token request over e in rounds r or $r+1$ is when v sends all its left token requests over its other new edges in rounds r or $r+1$. Then, due to 3-edge stability, v will receive its requested tokens by the end of round $r+2$ and becomes complete. Otherwise, v sends a token request over e in rounds r or $r+1$, and hence e becomes contributive by the end of round $r+2$.

Then, the only case when an edge becomes idle in round r, is where both the endpoints are incomplete in round $r-1$ and only one of them becomes complete in round r. Therefore, since each node v becomes complete once, the number of v's idle edges never increases throughout the execution after v's completion.

Now consider an arbitrary futile round where the largest number of idle edges of any complete node in a futile round is x. Hence, there exist at least x bridge nodes in that round. Therefore, based on Lemma 3.2, at least x idle edges are removed at the end of that futile round. As a result, one can see that there cannot be any idle edges, and hence any futile rounds, after having n futile rounds. This shows that the number of futile rounds is at most n until the last token request is sent. □

Theorem 3.4. *Given k tokens to disseminate, if the dynamic graph is 3-edge stable, the Single-Source Unicast Algorithm terminates in $O(nk)$ rounds and all the nodes receive all the k tokens.*

Proof. Consider any time t during an arbitrary execution of the Single-Source Algorithm that is not terminated yet. Let k' denote the number of token learnings in $[0, t]$. Let us show that the number of periods of two consecutive rounds in $[1, t]$ in which no token is learned is at most $k' + n$. This leads to $O(nk)$ running time for the algorithm.

Let r and $r+1$ be arbitrary two consecutive rounds in $[1, t]$, where no token is learned. Hence, there is no new edge in round $r-1$, otherwise, a token would have been learned in round r or $r+1$ due to the 3-edge stability property and the higher priority of sending token requests on new edges. Then, there are two possibilities:

- Case 1: At least one contributive edge carries a token request in round $r-1$. Since it is assumed that no token is learned in round r, the edge must be removed by the adversary at the end of round $r-1$. Therefore, we can map one of the removed contributive edges to round r. Doing so, for any such round r, a distinct token learning in $[0, t]$ is mapped to r (i.e., one of the token learnings that happened on the removed contributive edge after its last insertion). Therefore, since there is a one to one mapping between such rounds and a subset of token learnings in $[0, t]$, the number of such rounds (i.e., r) is not more than the number of token learnings in $[0, t]$.
- Case 2: No contributive edge carries a token request in round $r-1$. Therefore, round $r-1$ is a futile round. Then, based on Lemma 3.3, the number of such rounds (i.e., round r) is at most n throughout the execution.

□

3.2 Multiple Source Nodes

Let us consider a more general case where the tokens are initially given to more than one source node. Let us assume that there are s source nodes $a_1 < a_2 < \cdots < a_s$ such that for $1 \le i \le s$, a_i is initially given k_i tokens. Therefore, $k = \sum_i k_i$ tokens in total need to be disseminated to all the nodes in the network.

3.2.1 Strongly Adaptive Adversary

To solve this problem against a strongly adaptive adversary, we present a deterministic algorithm with $O(n^2 s + nk) + \mathsf{TC}(\mathcal{E})$ message complexity. It extends the Single-Source Unicast Algorithm, and has the same running time if the network has the same stability assumption (i.e., 3-edge stability).

Since there are more than one source nodes, we need to include the intended source node in the definitions of Section 3.1. So we say a node is complete *with respect to* source node a, if it has received all the tokens originated at a. Similarly, a node is called a bridge node *with respect to* source node a, if it is an incomplete node with respect to a and is connected to a node which is complete with respect to a.

Multi-Source Unicast Algorithm

The algorithm considers a priority over the dissemination of tokens from different sources. To do so, in each round, all nodes give the highest priority to the dissemination of the tokens from the minimum known source node whose dissemination is not yet complete. In the sequel, we explain the details of implementing this idea.

Initially, each source node x considers an arbitrary order of its tokens and assigns a token identifier containing its own ID and an integer i (i.e., $\langle ID_x, i\rangle$) to its i^{th} token. Moreover, we assume that each source node becomes complete with respect to itself at time 0. To avoid redundant communication, each node v keeps some information about the execution history by constantly updating the following sets. $R_v(x)$ is the set of all nodes that are informed by v about the v's completeness with respect to x. $S_v(x)$ is the set of nodes that informed v about their completeness with respect to x. I_v is the set of all source nodes with respect to which v is complete. Then each node v in each round of the execution does the following three tasks in parallel:

- For each edge $\{v, w\}$, if there is any source node x such that $x \in I_v$ and $w \notin R_v(x)$, it picks the minimum such x and sends "completeness announcement with respect to x" to w.
- For each edge $\{v, w\}$, if v received a request for token τ from w in the previous round, then it sends τ to w.
- Node v picks the minimum x such that $x \notin I(v)$ and $S_v(x) \neq \emptyset$. Then, regarding sending token requests, it acts similarly to the Single-Source Unicast Algorithm as there exists only the single source x in the network.

Theorem 3.5. *To disseminate k tokens which are initially distributed among s source nodes, Multi-Source Unicast Algorithm has a 1-adversary competitive message complexity of $O(n^2 s + nk)$.*

Proof. We argue similarly to the Proof 3.1.2. There are three different types of messages sent by nodes during the algorithm execution; (1) token, (2) completeness announcement, and (3) token request. Each node sends the request of each distinct token to at most one neighbor in a round. If the connection to that complete neighbor remains for the very next round, then the requested token will be successfully received by the node and the node stops sending this token request. Therefore, each distinct token is received by each node once, and we can conclude that there are at most $O(nk)$ sent messages of type 1 throughout the execution.

There are n nodes in the network, each informs at most $n-1$ other nodes about its completeness with respect to at most s source nodes throughout the execution. Since each node avoids informing the same node more than once, there are at most $O(n^2 s)$ messages of type 2 being sent throughout the execution.

It remains to show that the number of sent messages of type 3 is at most $O(nk) + \mathsf{TC}(\mathcal{E})$, where $\mathsf{TC}(\mathcal{E})$ is the number of topological changes during the execution (i.e., the number of edge insertions while considering the initial graph as an empty graph). In each round where a token request is sent by some node, a new token is received in the next round unless the edge is removed. Therefore, we can say that the number of token requests sent at any time is at most $O(nk)$ plus the number of edge deletions. $O(nk)$ comes from the fact that there exist k tokens and each token is received by at most $O(n)$ nodes, each token once. Further, since we assume that the initial graph is an empty graph, the number of edge deletion is upper bounded by $\mathsf{TC}(\mathcal{E})$. $\square$

Theorem 3.6. *Given k tokens to disseminate, if the dynamic graph is 3-edge stable Multi-Source Unicast Algorithm terminates in $O(nk)$ rounds and all the nodes have received all the k tokens.*

Proof. Theorem 3.4 states when all the k tokens are initially given to one source node, by running Single-Source Unicast Algorithm, k-token dissemination is complete in at most $O(nk)$ rounds. Multi-Source Unicast Algorithm guarantees that the minimum ID source node that its token dissemination is not complete yet runs the Single-Source Unicast Algorithm without any interference until its token dissemination is complete. It is guaranteed by having all the nodes giving the highest priority to the token dissemination of the the minimum ID source node with incomplete token dissemination.

Therefore, if the Single-Source Unicast Algorithm solves k-token dissemination in cnk rounds for some constant c, then the token dissemination of the first minimum ID source node is complete after cnk_1 rounds and the second one after the next cnk_2 rounds and so on. Hence, the whole running time is $O(nk)$, where $k = \sum_{i=1}^{s} k_i$.

□

3.2.2 Oblivious Adversary

In this section, we focus on instances with multiple source nodes where $o(n^2)$ tokens in total are distributed arbitrarily among s source nodes. Assume that the total number of tokens k is known to all the nodes. Note that when there are $\Omega(n^2)$ tokens, the algorithm against a strongly adaptive adversary describted in the previous section achieves optimal amortized message complexity. However, that algorithm fails to achieve an efficient message bound when there are $o(n^2)$ tokens and a large number of source nodes, e.g., more than $\Omega(n)$ sources. By weakening the adversary from a strongly adaptive adversary to an oblivious adversary, we next show how to achieve a $o(n^2)$ amortized message complexity (see the definition of adversaries in Section 1.3). The algorithm in this section builds on the previous Multi-Source-Unicast algorithm (cf. Section 3.2.1). As given by Theorem 3.5, when $s \gg n/k$, the dominating term in the message complexity of the Multi-Source-Unicast algorithm is $O(n^2 s)$, which comes from the fact that whenever a node becomes complete with respect to some source node, it informs all its neighbors throughout the execution. This essentially costs $O(n^3)$ messages if there are n sources as all the nodes have to do this for all the source nodes and in the worst-case for $\Omega(n)$ neighbors per source.

In the following, we assume that the number of source nodes is known to all the nodes in the network initially. Intuitively, the idea of the algorithm is to reduce the number of source nodes. Let us call the (reduced) source nodes the *centers*. The algorithm runs in two phases. In the first phase, it reduces the number of source nodes to some appropriate number of centers. All the k tokens are then distributed among the centers. Then in the second phase, it applies the Multi-Source-Unicast algorithm with the centers as new source nodes.

Let us now discuss how to reduce the source nodes to f centers (we fix the value of f later). If the number of source nodes is less than $n^{2/3} \log n$ then we set $f = s$ and we directly go to the second phase and run the Multi-Source-Unicast algorithm (cf. Section 3.2.1). We therefore in the following assume that $s \geq n^{2/3} \log n$ for the remainder of this section. Each node chooses itself as a center independently with probability f/n. Hence, in expectation, f centers are selected. Each source node (if it is not selected as a center) needs to send its tokens to some center. Multiple tokens from the same source node can go to multiple centers, however, a single token goes to only one center. Since there are f random centers, any fixed set of $O(n \log(n)/f)$ distinct nodes must have at least one center with high probability (whp)[8]. Hence, if a node has a degree more than $O(n \log(n)/f)$, then it has at least one center among its neighbors. As long as a node has a center as its neighbor, it can easily send its tokens (if any) to the center. Otherwise, the general idea is that all the tokens in the network perform a random walk (in parallel) until they reach some center. Once a token reaches a center, it stops there. Therefore, each random walk of a token has to visit $\Omega(n \log n/f)$ distinct nodes to guarantee that it hits a center whp. For this, we apply a known random walk visit bound (see Lemma 3.7 below) for the dynamic setting [16]. Suppose all the tokens reach some center. Each center then considers an arbitrary order of the tokens and assigns a

[8]'with high probability' (whp) means with probability at least $1 - \frac{1}{n}$.

token identifier containing its ID and the order number. Then they run the Multi-Source-Unicast algorithm (cf. Section 3.2.1) by considering the f centers as the new source nodes.

Analysis. Consider a random walk of a token. Assume that the random walk has to take L steps to visit $\Omega(n \log n/f)$ distinct nodes in the dynamic graph whp. If the random walk in some step τ visits a node v whose degree is more than $(n \log n)/f$ in round $\tau + 1$, then the walk stops there (whp) as in the next step the random walk will reach a center. The reason is that the degree of node larger than $(n \log n)/f$ means it has at least one center in the neighbor set whp. Hence, we put some conditions on the random walk. Let δ be this degree threshold i.e., $\delta = (n \log n)/f$. Given a specific graph sequence and a starting node v, a random walk is a distribution over deterministic walks. Let us call such a deterministic walk (of length L) "bad" if one of the following three conditions hold: the random walk (I) does not visit a node of degree more than δ and it does not hit one of the f centers, (II) arrives at a node v with degree more than δ, but v does not have a center as a neighbor, (III) before hitting the first high degree node (i.e., degree more than δ), the number of steps exceeds $2L$. If L steps assure visiting $\Omega(n \log(n)/f)$ distinct nodes whp, any random walk of length L will visit a center whp (and thus condition (I) has polynomially small probability). Further, if the walk arrives at a high-degree node, the probability that the nodes do not have a center in its neighborhood is also polynomially small. Finally, because all low-degree nodes have sufficiently many self loops (a $1 - O(f/\log n)$-fraction of their edges), also condition (III) only happens with polynomially small probability. We therefore get that a walk is good (i.e., non-bad) whp. Moreover, a good walk will end at one of the f centers.

Lemma 3.7 (Lemma 6.7 in [16])**.** *Let $\mathcal{G}$ be a d-regular dynamic graph controlled by an oblivious adversary. Let $N_x^t(y)$ denote the number of visits of a random walk to vertex y by time t, given that the random walk started at node x. $N_x^t(y)$ could be zero or positive number. Then for any nodes x, y and for all $t = O(\tau_{mix})$, where τ_{mix} is the mixing time of $\mathcal{G}$,*

$$\Pr\left(N_x^t(y) \geq 32d\sqrt{t+1}\log n\right) \leq \frac{1}{n^2}$$

The above lemma implies that a random walk of length L on a d-regular dynamic graph visits at least $L/(32d\sqrt{L+1}\log n)$ i.e., $\Omega(\sqrt{L}/d \log n)$ distinct nodes with high probability.

Let us construct a virtual regular multigraph by adding an appropriate number of self-loops to the network at each round. Note that if a node has actual degree δ, it has to add $n - \delta$ self-loops to make its degree n. We remark that these self-loops are virtual edges, only for the purpose of random walk. That is, if a node v has degree at most δ, with probability at least $(1 - \delta/n)$, the random walk goes to a self-loop and performs a virtual step. Otherwise with the remaining probability δ/n, it follows a real edge in the dynamic graph.

Suppose the random walk is good, i.e., none of the above three conditions ever happens. Then in order to perform L actual steps, the walk w.h.p. takes at least $\Omega(nL/\delta)$ steps on the n-regular multigraph. Therefore, the bound we get from Lemma 3.7 implies that the walk visited $\Omega(\sqrt{nL/\delta}/n \log n) = \Omega(\sqrt{L/(\delta n \log^2 n)})$ distinct nodes. Since the requirement is that the walk should visit at least $\Omega(n \log n/f)$ distinct nodes in this dynamic network, we get,

$$\sqrt{L/(\delta n \log^2 n)} \geq (n \log n)/f$$

Since the threshold degree $\delta = n \log n/f$, the length of the random walk is $L \geq n^4 \log^4 n/f^3$. This implies that each token performs a random walk of length at least $n^4 \log^4 n/f^3$ to guarantee that it reaches a center.

Message Complexity. Let us now calculate the message complexity of the algorithm by setting the appropriate values of the parameters. The following theorem holds when the number of source nodes is $\Omega(n^{1/2})$.

Theorem 3.8. *There is an algorithm with message complexity $O(n^{5/2}k^{1/4}\log n)$ to disseminate $k = o(n^2)$ tokens from $\Omega(n^{1/2})$ source nodes in a dynamic network, the topology of which is controlled by an oblivious adversary. Hence, the amortized message complexity of the algorithm is $O(n^{5/2}\log n/k^{3/4})$.*

Proof. In the first phase, at most k tokens perform random walks of (actual) L steps to reach some center node. Note that this excludes message cost for the self-loop edges. Therefore, it costs kL messages in the first phase. In the second phase, we run Multi-Source-Unicast algorithm with f source nodes. The message cost in the second phase is $O(fn^2 + nk)$, by Theorem 3.5. Thus the total message cost in both the phases is $O(kL+fn^2+nk)$. Since L is larger than n, $kL > kn$. The message bound becomes $O(kL+fn^2)$. Optimizing the sum $(kL + fn^2)$ one gets,

$$\begin{aligned}
& kL = fn^2 \\
\Rightarrow\, & L = fn^2/k \\
\Rightarrow\, & n^4 \log^4 n/f^3 = fn^2/k \qquad \text{[Substituting } L = n^4 \log^4 n/f^3\text{]} \\
\Rightarrow\, & f = n^{1/2}k^{1/4}\log n
\end{aligned}$$

Thus, the total message complexity is $O(fn^2) = O(n^{\frac{5}{2}}k^{\frac{1}{4}}\log n)$.

Therefore the amortized message complexity for k tokens is $O(n^{\frac{5}{2}}k^{\frac{1}{4}}\log n)/k = O\left(\frac{n^{\frac{5}{2}}\log n}{k^{\frac{3}{4}}}\right)$. □

Recall that we assumed the number of source nodes is more than $n^{\frac{2}{3}}\log n$, so the number of tokens is at least $n^{\frac{2}{3}}\log n$. The following chart highlights the amortized message cost for different sizes of the token set.

- $k = O(n)$, the amortized message cost is $O(n^{\frac{7}{4}}\log n)$ which is $o(n^2)$.
- $k = O(n^{\frac{3}{2}})$, the amortized message cost is $O(n^{\frac{11}{8}}\log n)$.
- $k = O(n^2)$, the amortized message cost is $O(n\log n)$.
- $k = O(n^{\frac{2}{3}}\log n)$, the amortized message cost is $O(n^2(\log n)^{\frac{1}{4}})$.

Consider the case when the number of source nodes is less than $n^{\frac{2}{3}}\log n$. As mentioned earlier, Multi-Source-Unicast algorithm is executed in this case. Note that Multi-Source-Unicast algorithm is deterministic and hence will work in the dynamic graph controlled by an oblivious adversary. The total message cost of Multi-Source-Unicast algorithm is $O(n^2 s + nk)$ (cf. Theorem 3.5). Thus the amortized message complexity is $O(\frac{n^2 s}{k} + n)$. Since the number of tokens is always larger than number of source nodes, $s/k \leq 1$. Hence, the amortized message cost is bounded by $O(n^2)$.

Running Time. The running time of the first phase is the time to perform random walks corresponding to all the tokens in parallel. Since the dynamic graph is n-regular, as many as $O(n^2)$ random walks can be executed in parallel with at most $O(\log n)$ congestion over an edge. The reason is that if each node starts $O(n)$ random walks in parallel, in expectation, each edge carries at most 2 walks (from both ends) in each round. Hence, there will be at most $O(\log n)$ congestion over an edge with high probability. Therefore, one can perform $O(n^2)$ random walks ($O(n)$ from each node) in parallel with at most $O(\log n)$ delay per step. Since the length of the random walks is $O(L)$, the total time of the first phase is thus $O(L\log n)$ rounds. Since the second phase is the execution of Multi-Source-Unicast algorithm, it takes $O(nk)$ time with the additional natural condition that the dynamic graph is 3-edge stable, see Theorem 3.6. Hence the total running time is $O(L\log n + nk)$ rounds, which is $O\left(n^{\frac{5}{2}}\log^4 n/k^{\frac{3}{4}} + nk\right)$, since $L = O\left(n^{\frac{5}{2}}\log^3 n/k^{\frac{3}{4}}\right)$.

4 Conclusion and Open Problems

We studied the message complexity of information spreading in dynamic networks. While time complexity has been studied more intensely, understanding the message complexity in various dynamic network models is likely to shed light on the time complexity as well. Several open questions arise from our work. One key question is that we do note have tight bounds on the amortized message complexity of unicast under the strongly adaptive adversary (when not charging the adversary for topological changes). The only known bounds are the trivial $O(n^3)$ upper and $\Omega(n)$ lower bounds.

A contribution of our work is introducing the adversary-competitive message complexity which is useful for studying algorithmic performance in dynamic networks as a function of the dynamism. We were able to show an optimal amortized message bound for unicast in this model for both the single-source and multi-source setting, when the number of tokens is large. However, when the number of tokens is small (say n) and they start from multiple sources (an important special case is one token starts from each node), we do not have a good bound. We were able to show only a $o(n^2)$ amortized bound under a weaker (oblivious) adversary. Improving this bound for oblivious adversary or showing a non-trivial bound for the strongly adaptive adversary is an interesting open problem.

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