An $O(n \log n)$-Space Decision Procedure for the Relevance Logic B^+ (Extended Version)*

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Abstract

In previous work we gave a new proof-theoretical method for establishing upper-bounds on the space complexity of the provability problem of modal and other propositional non-classical logics. Here we extend and refine these results to give an $O(n \log n)$-space decision procedure for the basic positive relevance logic B^+. We compute this upper-bound by first giving a sound and complete, cut-free, labelled sequent system for B^+, and then establishing bounds on the application of the rules of this system.

1 Introduction

Context This paper is part of ongoing research that aims at developing a framework for applying proof theory to the analysis of the complexity of families of non-classical logics, and thereby designing decision procedures for these logics [4, 5, 6, 35].

Modal, relevance and other non-classical logics are usually presented using Hilbert-style systems. However, although uniform, these systems are difficult to use in practice, especially in comparison with the more 'natural' Gentzen-style systems such as natural deduction, sequent and tableaux systems. Unfortunately, devising Gentzen-style systems for non-classical logics often requires considerable ingenuity, as well as trading uniformity for simplicity and usability.

A solution to the problem of formalizing and using non-classical logics is to employ *labelling* techniques, which provide a general framework for presenting different logics in a uniform way as Gentzen-style deduction systems. The intuition behind these techniques is that labelling (sometimes also called prefixing, annotating or subscripting) allows one to encode explicitly additional information, of a semantic or proof-theoretical nature, that is otherwise implicit in the logic one wants to formalize. So, for instance, instead of considering a modal formula A we consider the *labelled formula* $x{:}A$, where x denotes a possible world in the underlying Kripke semantics for modal logics. In other words, the label x keeps track of the possible world in which the formula A lives. We can then use labels also to specify they way in which possible worlds are related in a particular Kripke model, e.g. the binary *relational formula* $x\,R\,y$ tells us that the world y is accessible from x.

Based on the works of Anderson and Belnap [1, 2], Fitting [13] and Gabbay [14], among others, labelled deduction systems have been used extensively to formalize non-classical logics, and exploited for applications that can be modelled by these logics. For example, several labelled deduction systems have been given for relevance logics, e.g. [1, 2, 5, 8, 9, 13, 16, 20], ranging from systems where the labels explicitly encode information from the Kripke or algebraic semantics, to systems where the additional information is of a proof-theoretical nature, e.g. encoding resources and their use.

*Technical Report No. 140, May 2000, Institut für Informatik, Albert-Ludwigs-Universität Freiburg, Germany.

Research on labelling has focused not only on the design of labelled deduction systems for specific non-classical logics, but also, more generally, on the characterization of the classes of logics that can be formalized this way, as well as on investigations of the general properties and limitations of labelling techniques [3, 5, 10, 14, 17, 35].

Labelled deduction systems can also be used to establish the decidability of logics and analyze their complexity. For example, in [35] we use labelling to devise a new proof-theoretical method for establishing upper-bounds on the space complexity of the decision problem for the propositional modal logics K, T, K4 and S4. Our method consists of three steps. In the first step, we give sound and complete, cut-free, labelled sequent systems for propositional modal logics. The absence of cut is a first step towards decidability in that it yields a subformula property bounding the number of different formulas arising in proofs. To bound proof search, however, we must also bound how often subformulas might appear. In fact, some of the structural rules (i.e. the contraction rules) of our sequent systems duplicate formulas. Since these rules are always applicable, infinite branches may arise and proof search may not terminate. Thus, in the second step of our method, we analyze our modal labelled sequent systems and show how to restrict their structural rules; in other words, we establish bounds on the applications of the rules and use these bounds to define a measure that decreases at every rule application. In the third and final step, we combine these bounds with an analysis of the accessibility relation of the corresponding Kripke models to give decision procedures for K, T, K4 and S4, and compute polynomial upper-bounds on the space requirements of these procedures.

In [6] we generalize this method to a recipe for giving decision procedures for propositional non-classical logics, which we present using cut-free labelled sequent systems, and for establishing upper-bounds on their space complexity. Our recipe consists of guidelines and examples so that, given a particular logic, the user can contribute bounds on structural reasoning and the accessibility relation of the corresponding Kripke semantics. Among other examples, in [6] we briefly indicate that the implication-conjunction fragment of the relevance logic B admits a Polynomial Space decision procedure.

Contributions Our main contribution here is to give an $O(n \log n)$-space decision procedure for the basic positive relevance logic B^+ by introducing a sound and complete, cut-free, labelled sequent system $\mathrm{S}(\mathrm{B}^+)$, and investigating $\mathrm{S}(\mathrm{B}^+)$ to eliminate some of its rules or establish bounds on their applications. We view this as a two-fold contribution. First, our decision procedure and space upper-bound for B^+ are new. Second, this work constitutes not only an interesting case study but also, more concretely, a first step towards the application of our framework to the proof-theoretical investigation of the decidability and computational complexity of the relevance logics that can be obtained by extending the basic logic B^+. Indeed, our procedure extends the method of [35] and the results fit and refine the abstract recipe of [6]. It is however important to note that that recipe has a general character, while most of the results that we establish here follow by a refined analysis of $\mathrm{S}(\mathrm{B}^+)$; i.e. they could not have been captured simply by instantiating the guidelines of our recipe to B^+.

Related Work Our bound is new. To our knowledge, no similar investigations have been carried out on relevance logics, and, in particular, we are not aware of any subrecursive bounds for the basic relevance logic B or its fragments. In fact, the analysis of decidability and complexity of relevance logics is a subtle issue (especially in comparison with other non-classical logics such as modal logics), and positive results have been presented alongside open problems and negative results, including the undecidability of the principal relevance logics R and E, e.g. [1, 2, 25, 26, 29, 31, 32, 33].

The decidability of relevance logics is often established semantically by defining filtrations on possible worlds and proving that the logics possess the finite model property, e.g. [1, 2, 26, 29, 32]. Although effective to establish decidability, this often yields poor complexity bounds. Moreover, as remarked in, e.g., [25], the standard proofs of the finite model property fail for several relevance logics, although not for B and its fragments.

Proof-theoretical decision procedures based on Gentzen-style systems have also been given for B and other relevance logics, e.g. [1, 2, 7, 15, 16, 25, 26]. Although we are not aware of any analysis of the complexity of these procedures, some of this work is close to ours, at least in spirit.

In [16], Gochet, Gribomont and Rossetto extend the *connection method* [36] to B^+, and give a decision procedure that yields finite models for satisfiable formulas. However, this method investigates the need for

duplications, via explicit and implicit contractions, of indexed formulas, rather than explicitly bounding applications of the contraction rules as we do here. (We return to this below.)

In [25], Restall gives *display calculi* for a wide range of substructural logics, and investigates their decidability by a proof-theoretical analysis. Similar to ours, this analysis focuses on controlling applications of the contraction rules. However, Restall must also control applications of other rules, which are specific rules of display calculi, such as the intensional display rules. (A similar analysis is required by the Gentzen-style systems of [1, 7, 15], which are also based on the introduction of new intensional connectives.) While the display method establishes the decidability of several substructural logics, including the logic DW, which is a simple extension of B, no complexity results are given in [25]. Investigating the connections and complementary aspects of Restall's approach and ours remains as future work.

Organization The remainder of this paper is organized as follows. In § 2 we introduce our cut-free labelled sequent system $\mathrm{S(B^+)}$ and show that it is sound and complete with respect to the corresponding Kripke semantics. In § 3 and § 4, the core of the paper, we bound the space complexity of proof search in $\mathrm{S(B^+)}$. We first prove some preliminary results, § 3.1, and then, in § 3.2–3.4, show that we can bound or eliminate altogether applications of some of the rules of $\mathrm{S(B^+)}$. In particular, Theorem 3.22 shows that we can eliminate all contractions in proofs of theorems of $\mathrm{S(B^+)}$. Based on this analysis, in § 4 we then give our decision procedure for $\mathrm{S(B^+)}$ and establish an $O(n \log n)$ upper-bound on its space complexity (Theorem 4.2). Finally, in § 5 we draw conclusions and discuss future work.

2 The Sequent System $\mathrm{S(B^+)}$

We now introduce our cut-free labelled sequent system $\mathrm{S(B^+)}$; we assume the reader is familiar with the syntax and semantics of relevance logics, e.g. [2, 29], and with Gentzen-style systems such as sequent and natural deduction systems, e.g. [22, 30].

We use labelling to encode explicitly semantic information. To illustrate the labelling technique that we employ, consider the valuation clause for relevant implication $\to$ in a Kripke model $\mathfrak{M}$ for positive relevance logics. Let $\mathfrak{M} = (\mathfrak{W}, \mathfrak{R}, \mathfrak{V})$, where $\mathfrak{W}$ is a non-empty set of possible worlds, including the *actual world* 0, $\mathfrak{R}$ is a ternary (*compossibility*) relation on $\mathfrak{W}$, and $\mathfrak{V}$ is a function mapping elements of $\mathfrak{W}$ and propositional variables to truth values (0 or 1), based on which we define the truth relation $x \vDash A$ for $x \in \mathfrak{W}$ and A a formula of $\mathrm{B^+}$. In particular, given a model $\mathfrak{M}$, the truth of $A \to B$ at a world x is evaluated in terms of the truth of A at y and of B at z where $(x, y, z) \in \mathfrak{R}$, i.e.

$$x \vDash A \to B \text{ iff for all } y, z \in \mathfrak{W}, \text{ if } (x, y, z) \in \mathfrak{R} \text{ and } y \vDash A \text{ then } z \vDash B.$$

Now let W be a denumerable set of *labels* $x, y, z, a, b, \ldots$, and R a ternary relation over W. In our labelled sequent system $\mathrm{S(B^+)}$ we consider two kinds of formulas: *labelled well-formed formulas* (*lwffs*), written $x{:}A$, which pair a label x and a formula A of $\mathrm{B^+}$, and *relational well-formed formulas* (*rwffs*), written $R\,x\,y\,z$. The lwff $x{:}A$ expresses that the formula A holds at world x, and the rwff $R\,x\,y\,z$ expresses that (x, y, z) is in the relation $\mathfrak{R}$.[1] We can then modify the above semantic definition to suit this extended language as follows:

$$\vDash x{:}A \to B \text{ iff for all } y, z \in \mathfrak{W}, \text{ if } \vDash R\,x\,y\,z \text{ and } \vDash y{:}A \text{ then } \vDash z{:}B,$$

where the relation $\vDash$ now expresses truth for labelled and relational formulas.

This semantics can be straightforwardly captured by the sequent rules

$$\frac{\Delta \vdash R\,x\,y\,z \quad \Gamma, \Delta \vdash \Gamma', y{:}A \quad z{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \to B, \Gamma, \Delta \vdash \Gamma'} \to\mathrm{L} \qquad \frac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B} \to\mathrm{R},$$

where (i) Γ and Δ, possibly annotated, vary over finite multisets of lwffs and rwffs respectively; and (ii) application of $\to$R is subject to the condition that y and z are distinct and do not occur in $\Gamma, \Delta \vdash \Gamma', x{:}A \to B$.

Two remarks. First, following [30], we call the formulas that pass through the application of a rule unchanged (they appear in the premises and in the conclusion) the *parametric formulas* of the rule; the

[1] Note that to reduce notational overhead, we do not define an interpretation function mapping labels into worlds in $\mathfrak{W}$ but implicitly identify the label x with the world $x \in \mathfrak{W}$, i.e. we identify W with $\mathfrak{W}$; cf. the discussion in [35].

formula contracted or introduced in the conclusion (on the left of the sequent symbol $\vdash$, in symbols ... L, or on its right, ... R) is the *principal formula* of the rule, and the formulas from which the principal formula derives are the *active formulas* of the rule. Second, our sequents have the form $\Gamma, \Delta \vdash \Gamma'$ or $\Delta \vdash R\,x\,y\,z$, i.e. we have multi-conclusioned sequents for lwffs and single-conclusioned sequents for rwffs. In [5, 6, 35] we show that if we adopt these distinct forms for sequents, then we are able to formalize non-classical logics whose Kripke semantics is axiomatizable in terms of Horn clauses, including many of the most common logics. Most importantly, sequents of these forms allow us to devise cut-free sequent systems for these logics and provide the basis for the complexity analysis that we perform below.[2]

The other connectives of $\mathrm{S(B^+)}$ are classical ones: conjunction $\wedge$ and disjunction $\vee$. They are evaluated locally, at the same world, e.g. $\vDash x{:}A \vee B$ iff $\vDash x{:}A$ or $\vDash x{:}B$. To reason about $\wedge$ and $\vee$, we simply adapt traditional sequent rules [22, 30] by adding labels and parametric rwffs, i.e.

$$\frac{x{:}A, x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \wedge B, \Gamma, \Delta \vdash \Gamma'}\ \wedge\mathrm{L}, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A \quad \Gamma, \Delta \vdash \Gamma', x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \wedge B}\ \wedge\mathrm{R},$$

$$\frac{x{:}A, \Gamma, \Delta \vdash \Gamma' \quad x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \vee B, \Gamma, \Delta \vdash \Gamma'}\ \vee\mathrm{L}, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \vee B}\ \vee\mathrm{R}.$$

The rules for $\wedge$, $\vee$ and $\rightarrow$ define the behavior of the connectives, and we call them *logical rules*. To define the system $\mathrm{S(B^+)}$, we also need axioms and additional rules for lwffs and rwffs (in symbols ...l... and ...r...). The *axioms*

$$\frac{}{x{:}A \vdash x{:}A}\ \mathrm{AXl} \quad \text{and} \quad \frac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\ \mathrm{AXr}$$

allow us to conclude backwards proofs, and the *structural rules*

$$\frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta, R\,x\,y\,z \vdash \Gamma'}\ \mathrm{WrL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\ \mathrm{WlL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma', x{:}A}\ \mathrm{WlR},$$

$$\frac{\Delta, R\,a\,b\,c, R\,a\,b\,c \vdash R\,x\,y\,z}{\Delta, R\,a\,b\,c \vdash R\,x\,y\,z}\ \mathrm{CrL}, \qquad \frac{x{:}A, x{:}A, \Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\ \mathrm{ClL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}A}{\Gamma, \Delta \vdash \Gamma', x{:}A}\ \mathrm{ClR},$$

operate on the structure of sequents, deleting formulas (*weakening*, W...) or duplicating them (*contraction*, C...). Note that for rwffs we do not need structural rules that operate on the right of $\vdash$, as we work with single-conclusioned sequents $\Delta \vdash R\,x\,y\,z$.

Relevance logics require the truth relation to be monotonic with respect to a *partial order* $\sqsubseteq$ *on worlds*, which is usually defined in terms of $\mathfrak{R}$ and the actual world 0, i.e. $x \sqsubseteq y$ iff $(0, x, y) \in \mathfrak{R}$ [2, 29]. The monotony of $\vDash$ is expressed by the following postulates, which hold for any worlds a, x, y and z, and for any formula A:

- if $\vDash a \sqsubseteq x$ and $\vDash R\,x\,y\,z$, then $\vDash R\,a\,y\,z$;
- if $\vDash a \sqsubseteq y$ and $\vDash R\,x\,y\,z$, then $\vDash R\,x\,a\,z$;
- if $\vDash z \sqsubseteq a$ and $\vDash R\,x\,y\,z$, then $\vDash R\,x\,y\,a$; and
- if $\vDash x \sqsubseteq y$ and $\vDash x{:}A$, then $\vDash y{:}A$.

In $\mathrm{S(B^+)}$, we define $\sqsubseteq$ in terms of R and 0, i.e. $x \sqsubseteq y$ iff $R\,0\,x\,y$, and formalize monotony in $\mathrm{S(B^+)}$ in terms of the *monotony rules*

$$\frac{\Delta \vdash R\,0\,a\,x \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,a\,y\,z}\ \mathit{monr1}, \qquad \frac{\Delta \vdash R\,0\,a\,y \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,x\,a\,z}\ \mathit{monr2},$$

$$\frac{\Delta \vdash R\,0\,z\,a \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,x\,y\,a}\ \mathit{monr3}, \qquad \frac{\Delta \vdash R\,0\,x\,y \quad \Gamma, \Delta \vdash \Gamma', x{:}A}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ \mathit{monl}.$$

[2] Note also that the forms of our sequents distinguish our approach from traditional *semantic embeddings*, in which the Kripke semantics is used to translate non-classical logics into first-order or higher-order logic; cf. the discussion in [5, 35].

$$\frac{}{x{:}A \vdash x{:}A}\ \mathrm{AXl}, \qquad \frac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\ \mathrm{AXr},$$

$$\frac{x{:}A, x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \wedge B, \Gamma, \Delta \vdash \Gamma'}\ \wedge\mathrm{L}, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A \quad \Gamma, \Delta \vdash \Gamma', x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \wedge B}\ \wedge\mathrm{R},$$

$$\frac{x{:}A, \Gamma, \Delta \vdash \Gamma' \quad x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \vee B, \Gamma, \Delta \vdash \Gamma'}\ \vee\mathrm{L}, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \vee B}\ \vee\mathrm{R},$$

$$\frac{\Delta \vdash R\,x\,y\,z \quad \Gamma, \Delta \vdash \Gamma', y{:}A \quad z{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \to B, \Gamma, \Delta \vdash \Gamma'}\ \to\mathrm{L}, \qquad \frac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B}\ \to\mathrm{R},$$

$$\frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta, R\,x\,y\,z \vdash \Gamma'}\ \mathrm{WrL}, \qquad \frac{\Delta, R\,a\,b\,c, R\,a\,b\,c \vdash R\,x\,y\,z}{\Delta, R\,a\,b\,c \vdash R\,x\,y\,z}\ \mathrm{CrL},$$

$$\frac{\Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\ \mathrm{WlL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma', x{:}A}\ \mathrm{WlR},$$

$$\frac{x{:}A, x{:}A, \Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\ \mathrm{ClL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}A}{\Gamma, \Delta \vdash \Gamma', x{:}A}\ \mathrm{ClR},$$

$$\frac{\Delta \vdash R\,0\,a\,x \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,a\,y\,z}\ \mathit{monr1}, \qquad \frac{\Delta \vdash R\,0\,a\,y \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,x\,a\,z}\ \mathit{monr2},$$

$$\frac{\Delta \vdash R\,0\,z\,a \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,x\,y\,a}\ \mathit{monr3}, \qquad \frac{\Delta \vdash R\,0\,x\,y \quad \Gamma, \Delta \vdash \Gamma', x{:}A}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ \mathit{monl},$$

$$\frac{}{\vdash R\,0\,x\,x}\ \mathit{iden}.$$

In $\to$R, y and z are distinct and do not occur in $\Gamma, \Delta \vdash \Gamma', x{:}A \to B$.

Figure 1: The cut-free labelled sequent system S(B$^+$)

Transitivity of the partial order $\sqsubseteq$ is then a special case of *monr3* when x is 0. Reflexivity of $\sqsubseteq$ corresponds to postulating that the relation $\mathfrak{R}$ satisfies the *identity property*, i.e. $(0, x, x) \in \mathfrak{R}$ for all x. In S(B$^+$), we formalize the identity of R with the rule

$$\frac{}{\vdash R\,0\,x\,x}\ \mathit{iden}.$$

Definition 2.1 We call *relational rule* a rule that has an rwff as its principal formula. The system S(B$^+$), which is given in Figure 1, then consists of: the axioms AXl and AXr; the logical rules $\wedge$L, $\wedge$R, $\vee$L, $\vee$R, $\to$L and $\to$R; the structural rules WrL, WlL, WlR, CrL, ClL and ClR; the relational monotony rules *monr1*, *monr2* and *monr3*; the monotony rule for lwffs *monl*; and the relational rule *iden*. ■

That our cut-free system S(B$^+$) is a sound and complete system for the logic B$^+$ follows from the results in [5, 35]. In particular, there we introduced a framework for giving labelled natural deduction systems for various non-classical logics, and showed that such systems are sound and complete with respect to the corresponding Kripke semantics. Moreover, for the logics whose Kripke semantics is axiomatizable in terms of Horn clauses, we showed that derivations in our natural deduction systems normalize. That is, derivations can be reduced to a structured normal form where there are no unnecessary detours.

Instantiating that framework to give a sound and complete, normalizing, labelled natural deduction system N(B$^+$) for B$^+$ is a straightforward task, and we give N(B$^+$) in Figure 2 and point the reader to [35, p. 65] without further comment. The soundness and completeness of S(B$^+$) follows then from the general

$$\frac{x{:}A \wedge B}{x{:}A}\ \wedge\mathrm{El}, \quad \frac{x{:}A \wedge B}{x{:}B}\ \wedge\mathrm{Er}, \quad \frac{x{:}A \quad x{:}B}{x{:}A \wedge B}\ \wedge\mathrm{I}, \quad \frac{x{:}A \vee B \quad \begin{array}{c}[x{:}A]\\ \vdots\\ y{:}C\end{array} \quad \begin{array}{c}[x{:}B]\\ \vdots\\ y{:}C\end{array}}{y{:}C}\ \vee\mathrm{E},$$

$$\frac{x{:}B}{x{:}A \vee B}\ \vee\mathrm{Il}, \quad \frac{x{:}A}{x{:}A \vee B}\ \vee\mathrm{Ir}, \quad \frac{x{:}A \to B \quad y{:}A \quad R\,x\,y\,z}{z{:}B}\ \to\mathrm{E}, \quad \frac{\begin{array}{c}[y{:}A]\ [R\,x\,y\,z]\\ \vdots\\ z{:}B\end{array}}{x{:}A \to B}\ \to\mathrm{I},$$

$$\frac{R\,0\,a\,x \quad R\,x\,y\,z}{R\,a\,y\,z}\ monr1, \quad \frac{R\,0\,a\,y \quad R\,x\,y\,z}{R\,x\,a\,z}\ monr2, \quad \frac{R\,0\,z\,a \quad R\,x\,y\,z}{R\,x\,y\,a}\ monr3,$$

$$\frac{x{:}A \quad R\,0\,x\,y}{y{:}A}\ monl, \quad \frac{}{R\,0\,x\,x}\ iden.$$

In $\to$I, y and z are both fresh.

Figure 2: The natural deduction system N(B^+)

theorem of [35], which shows that normalizing labelled natural deduction systems and cut-free labelled sequent systems are intertranslatable. That is, we have:

Theorem 2.2 S(B^+) is sound and complete with respect to the corresponding Kripke semantics. In particular, the sequent $\vdash 0{:}A$ is provable in S(B^+) iff $\vDash 0{:}A$ (iff the formula A is valid in the Kripke semantics for B^+). ■

This follows by a straightforward adaptation and extension of Prawitz's results [22, App. A] for unlabelled deduction systems to suit labelled and relational formulas. We sketch the main proof ideas. Let us write $\vdash_{\mathrm{N}(\mathrm{B}^+)} x{:}A$ to indicate that the lwff $x{:}A$ is provable in N(B^+). Since by the results of [35] we know that $\vdash_{\mathrm{N}(\mathrm{B}^+)} 0{:}A$ iff $\vDash 0{:}A$, we need to show that the sequent $\vdash 0{:}A$ is provable in S(B^+) iff $\vdash_{\mathrm{N}(\mathrm{B}^+)} 0{:}A$. (We also need to show that the sequent $\vdash R\,x\,y\,z$ is provable in S(B^+) iff $\vdash_{\mathrm{N}(\mathrm{B}^+)} R\,x\,y\,z$.) The left-to-right direction follows by induction on the length of proofs of sequents in S(B^+). The right-to-left direction follows by induction on the length of normal derivations in N(B^+).

2.1 Some Example Proofs

We conclude this section with some examples of S(B^+)-proofs. We first prove the sequent $\vdash 0{:}((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$, which corresponds to one of the axioms of B^+ given in [28].

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{x{:}A \to C, x{:}B \to C, z{:}A, \Delta \vdash w{:}C} \quad \dfrac{\Pi_2}{x{:}A \to C, x{:}B \to C, z{:}B, \Delta \vdash w{:}C}}{x{:}A \to C, x{:}B \to C, z{:}A \vee B, \Delta \vdash w{:}C}\ \vee\mathrm{L}}{x{:}(A \to C) \wedge (B \to C), z{:}A \vee B, \Delta \vdash w{:}C}\ \wedge\mathrm{L}}{x{:}(A \to C) \wedge (B \to C), R\,0\,x\,y \vdash y{:}(A \vee B) \to C}\ \to\mathrm{R}}{\vdash 0{:}((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)}\ \to\mathrm{R}$$

where $\Delta = \{R\,0\,x\,y, R\,y\,z\,w\}$, and Π_1 is

$$\dfrac{\dfrac{\Pi_3}{\Delta \vdash R\,x\,z\,w} \quad \dfrac{\dfrac{\dfrac{}{z{:}A \vdash z{:}A}\ \mathrm{AXl}}{\vdots\ (\mathrm{W})}}{x{:}B \to C, z{:}A, \Delta \vdash w{:}C, z{:}A} \quad \dfrac{\dfrac{\dfrac{}{w{:}C \vdash w{:}C}\ \mathrm{AXl}}{\vdots\ (\mathrm{W})}}{w{:}C, x{:}B \to C, z{:}A, \Delta \vdash w{:}C}}{x{:}A \to C, x{:}B \to C, z{:}A, \Delta \vdash w{:}C}\ \to\mathrm{L}\ ,$$

Π_2 is

$$\dfrac{\begin{array}{ccc}\Pi_3 & \dfrac{}{z{:}B \vdash z{:}B}\ \mathrm{AXl} & \dfrac{}{w{:}C \vdash w{:}C}\ \mathrm{AXl} \\ \vdots & \vdots\ (\mathrm{W}) & \vdots\ (\mathrm{W}) \\ \Delta \vdash R\,x\,z\,w & x{:}A \to C, z{:}B, \Delta \vdash w{:}C, z{:}B & w{:}C, x{:}A \to C, z{:}B, \Delta \vdash w{:}C\end{array}}{x{:}A \to C, x{:}B \to C, z{:}B, \Delta \vdash w{:}C}\ {\to}\mathrm{L}$$

and Π_3 is

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\ \mathrm{AXr}}{\Delta \vdash R\,0\,x\,y}\ \mathrm{WrL} \quad \dfrac{\dfrac{}{R\,y\,z\,w \vdash R\,y\,z\,w}\ \mathrm{AXr}}{\Delta \vdash R\,y\,z\,w}\ \mathrm{WrL}}{\Delta \vdash R\,x\,z\,w}\ \mathit{monr1}\ .$$

Note that, for brevity, we denote a sequence of applications of the rule r in a derivation by vertical dots labelled with r, e.g.

$$\begin{array}{c}\Delta_2, R\,x\,y\,z \vdash R\,x\,y\,z \\ \vdots\ \mathrm{WrL} \\ \Delta_1, \Delta_2, R\,x\,y\,z \vdash R\,x\,y\,z\end{array}\ .$$

More generally, as we have done above, we denote an arbitrary number of applications of possibly different weakening rules by

$$\begin{array}{c}\Gamma_1, \Delta_1 \vdash \Gamma_1' \\ \vdots\ (\mathrm{W}) \\ \Gamma_1, \Gamma_2, \Delta_1, \Delta_2 \vdash \Gamma_1', \Gamma_2'\end{array}\ .$$

As further examples, we give the proofs of some of the other axioms of B^+ given in [28].

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\ \mathrm{AXr} \quad \dfrac{\dfrac{}{x{:}A \vdash x{:}A}\ \mathrm{AXl}}{x{:}A, R\,0\,x\,y \vdash x{:}A}\ \mathrm{WrL}}{x{:}A, R\,0\,x\,y \vdash y{:}A}\ \mathit{monl}}{\vdash 0{:}A \to A}\ {\to}\mathrm{R}\ .$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\ \mathrm{AXr} \quad \dfrac{\dfrac{\dfrac{}{x{:}A \vdash x{:}A}\ \mathrm{AXl}}{x{:}A, x{:}B \vdash x{:}A}\ \mathrm{WlL}}{x{:}A, x{:}B, R\,0\,x\,y \vdash x{:}A}\ \mathrm{WrL}}{x{:}A, x{:}B, R\,0\,x\,y \vdash y{:}A}\ \mathit{monl}}{x{:}A \wedge B, R\,0\,x\,y \vdash y{:}A}\ {\wedge}\mathrm{L}}{\vdash 0{:}(A \wedge B) \to A}\ {\to}\mathrm{R}\ .$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{ccc}\Pi_1 & & \Pi_2 \\ x{:}A \to B, x{:}A \to C, z{:}A, \Delta \vdash w{:}B & & x{:}A \to B, x{:}A \to C, z{:}A, \Delta \vdash w{:}C\end{array}}{x{:}A \to B, x{:}A \to C, z{:}A, \Delta \vdash w{:}B \wedge C}\ {\wedge}\mathrm{R}}{x{:}(A \to B) \wedge (A \to C), z{:}A, \Delta \vdash w{:}B \wedge C}\ {\wedge}\mathrm{L}}{x{:}(A \to B) \wedge (A \to C), R\,0\,x\,y \vdash y{:}A \to (B \wedge C)}\ {\to}\mathrm{R}}{\vdash 0{:}((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))}\ {\to}\mathrm{R}$$

where $\Delta = \{R\,0\,x\,y, R\,y\,z\,w\}$, Π_1 is

$$\dfrac{\begin{array}{ccc}\Pi_3 & \dfrac{}{z{:}A \vdash z{:}A}\ \mathrm{AXl} & \dfrac{}{w{:}B \vdash w{:}B}\ \mathrm{AXl} \\ \vdots & \vdots\ (\mathrm{W}) & \vdots\ (\mathrm{W}) \\ R\,0\,x\,y, R\,y\,z\,w \vdash R\,x\,z\,w & x{:}A \to C, z{:}A, \Delta \vdash w{:}B, z{:}A & w{:}B, x{:}A \to C, z{:}A, \Delta \vdash w{:}B\end{array}}{x{:}A \to B, x{:}A \to C, z{:}A, \Delta \vdash w{:}B}\ {\to}\mathrm{L}\ ,$$

Π_2 is

$$\dfrac{\begin{array}{ccc} \Pi_3 & & \dfrac{}{z{:}A \vdash z{:}A}\,\mathrm{AXl} \\ & & \vdots\,(\mathrm{W}) \\ R\,0\,x\,y, R\,y\,z\,w \vdash R\,x\,z\,w & x{:}A \to B, z{:}A, \Delta \vdash w{:}C, z{:}A & w{:}C, x{:}A \to B, z{:}A, \Delta \vdash w{:}C \end{array}}{x{:}A \to B, x{:}A \to C, z{:}A, \Delta \vdash w{:}C}\,{\to}\mathrm{L}$$

and Π_3 is

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\,\mathrm{AXr}}{R\,0\,x\,y, R\,y\,z\,w \vdash R\,0\,x\,y}\,\mathrm{WrL} \quad \dfrac{\dfrac{}{R\,y\,z\,w \vdash R\,y\,z\,w}\,\mathrm{AXr}}{R\,0\,x\,y, R\,y\,z\,w \vdash R\,y\,z\,w}\,\mathrm{WrL}}{R\,0\,x\,y, R\,y\,z\,w \vdash R\,x\,z\,w}\,\mathit{monr1}\ .$$

$$\dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\,\mathrm{AXr} \quad \dfrac{\dfrac{\dfrac{}{x{:}A \vdash x{:}A}\,\mathrm{AXl}}{x{:}A, R\,0\,x\,y \vdash x{:}A}\,\mathrm{WrL}}{x{:}A, R\,0\,x\,y \vdash x{:}A, y{:}B}\,\mathrm{WlR}}{\dfrac{\dfrac{x{:}A, R\,0\,x\,y \vdash y{:}A, y{:}B}{x{:}A, R\,0\,x\,y \vdash y{:}A \vee B}\,{\vee}\mathrm{R}}{\vdash 0{:}A \to (A \vee B)}\,{\to}\mathrm{R}}\,\mathit{monl}\ .$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc} \Pi_1 & \Pi_2 \\ x{:}A, x{:}B, R\,0\,x\,y \vdash y{:}A \wedge B, y{:}A \wedge C & x{:}A, x{:}C, R\,0\,x\,y \vdash y{:}A \wedge B, y{:}A \wedge C \end{array}}{x{:}A, x{:}B \vee C, R\,0\,x\,y \vdash y{:}A \wedge B, y{:}A \wedge C}\,{\vee}\mathrm{L}}{x{:}A \wedge (B \vee C), R\,0\,x\,y \vdash y{:}A \wedge B, y{:}A \wedge C}\,{\wedge}\mathrm{L}}{x{:}A \wedge (B \vee C), R\,0\,x\,y \vdash y{:}(A \wedge B) \vee (A \wedge C)}\,{\vee}\mathrm{R}}{\vdash 0{:}(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))}\,{\to}\mathrm{R}$$

where Π_1 and Π_2 are shown in (1) and (2).

As a final example, in (3) we prove the sequent

$$0{:}A \to B, 0{:}C \to D \vdash 0{:}(B \to C) \to (A \to D)\,,$$

which corresponds to the *affixing* rule

$$\dfrac{A \to B \quad C \to D}{(B \to C) \to (A \to D)}$$

of Hilbert-style axiomatizations of B^+.

3 Bounding Proof Search in $\mathrm{S}(\mathrm{B}^+)$

As is usual in sequent systems, we build $\mathrm{S}(\mathrm{B}^+)$-proofs by applying the rules *backwards*, starting with the end-sequent and working towards the axioms. In order to bound the complexity of proof search in $\mathrm{S}(\mathrm{B}^+)$ we then need to bound which formulas can appear in proofs and with how many repetitions.

To illustrate our analysis in more detail, we begin by observing that the absence of cut provides a first step towards decidability, in that it implies that $\mathrm{S}(\mathrm{B}^+)$ has a *subformula property*.

Definition 3.1 We say that $y{:}B$ is a *labelled subformula* of $x{:}A$ iff B is a subformula of A, i.e. iff A is B, or A is $A_1 \bullet A_2$ with $\bullet \in \{\wedge, \vee, \to\}$ and B is a subformula of A_1 or A_2. ■

Fact 3.2 [*Subformula property*] In any $\mathrm{S}(\mathrm{B}^+)$-proof of a sequent $\Gamma, \Delta \vdash \Gamma'$, only labelled subformulas of Γ and Γ' occur. ■

$$\dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\ \mathrm{AXr} \quad \dfrac{\dfrac{}{x{:}A \vdash x{:}A}\ \mathrm{AXl}}{\vdots\ \mathrm{(W)}}\ x{:}A, x{:}B, R\,0\,x\,y \vdash x{:}A, y{:}A \wedge C}{x{:}A, x{:}B, R\,0\,x\,y \vdash y{:}A, y{:}A \wedge C}\ \mathit{monl} \quad \dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\ \mathrm{AXr} \quad \dfrac{\dfrac{}{x{:}B \vdash x{:}B}\ \mathrm{AXl}}{\vdots\ \mathrm{(W)}}\ x{:}A, x{:}B, R\,0\,x\,y \vdash x{:}B, y{:}A \wedge C}{x{:}A, x{:}B, R\,0\,x\,y \vdash y{:}B, y{:}A \wedge C}\ \mathit{monl}}{x{:}A, x{:}B, R\,0\,x\,y \vdash y{:}A \wedge B, y{:}A \wedge C}\ \wedge\mathrm{R} \quad . \tag{1}$$

$$\dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\ \mathrm{AXr} \quad \dfrac{\dfrac{}{x{:}A \vdash x{:}A}\ \mathrm{AXl}}{\vdots\ \mathrm{(W)}}\ x{:}A, x{:}C, R\,0\,x\,y \vdash y{:}A \wedge B, x{:}A}{x{:}A, x{:}C, R\,0\,x\,y \vdash y{:}A \wedge B, y{:}A}\ \mathit{monl} \quad \dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\ \mathrm{AXr} \quad \dfrac{\dfrac{}{x{:}C \vdash x{:}C}\ \mathrm{AXl}}{\vdots\ \mathrm{(W)}}\ x{:}A, x{:}C, R\,0\,x\,y \vdash x{:}A \wedge B, x{:}C}{x{:}A, x{:}C, R\,0\,x\,y \vdash y{:}A \wedge B, y{:}C}\ \mathit{monl}}{x{:}A, x{:}C, R\,0\,x\,y \vdash y{:}A \wedge B, y{:}A \wedge C}\ \wedge\mathrm{R} \quad . \tag{2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash R\,0\,z\,z}\ \mathit{iden}}{\vdots\ \mathrm{WrL}}\ R\,0\,x\,y, R\,y\,z\,w \vdash R\,0\,z\,z \quad 0{:}C \to D, x{:}B \to C, z{:}A, R\,0\,x\,y, R\,y\,z\,w \vdash w{:}D, z{:}A \quad \dfrac{\dfrac{}{z{:}A \vdash z{:}A}\ \mathrm{AXl}}{\vdots\ \mathrm{(W)}}\ z{:}B, 0{:}C \to D, x{:}B \to C, z{:}A, R\,0\,x\,y, R\,y\,z\,w \vdash w{:}D}{0{:}A \to B, 0{:}C \to D, x{:}B \to C, z{:}A, R\,0\,x\,y, R\,y\,z\,w \vdash w{:}D}\ \to\mathrm{L}}{0{:}A \to B, 0{:}C \to D, x{:}B \to C, R\,0\,x\,y \vdash y{:}A \to D}\ \to\mathrm{R}}{0{:}A \to B, 0{:}C \to D \vdash 0{:}(B \to C) \to (A \to D)}\ \to\mathrm{R} \tag{3}$$

where Π_1 is

$$\dfrac{\dfrac{}{\vdash R\,0\,w\,w}\ \mathit{iden}}{\vdots\ \mathrm{WrL}}\ R\,0\,x\,y, R\,y\,z\,w \vdash R\,0\,w\,w \quad \Pi_2 \quad \dfrac{\dfrac{}{w{:}D \vdash w{:}D}\ \mathrm{AXl}}{\vdots\ \mathrm{(W)}}\ z{:}B, z{:}B \to C, z{:}A, R\,0\,x\,y, R\,y\,z\,w \vdash w{:}D, w{:}C \quad w{:}D, z{:}B, x{:}B \to C, z{:}A, R\,0\,x\,y, R\,y\,z\,w \vdash w{:}D}{z{:}B, 0{:}C \to D, x{:}B \to C, z{:}A, R\,0\,x\,y, R\,y\,z\,w \vdash w{:}D}\ \to\mathrm{L}$$

where Π_2 is

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\ \mathrm{AXr}}{R\,0\,x\,y, R\,y\,z\,w \vdash R\,0\,x\,y}\ \mathrm{WrL} \quad \dfrac{\dfrac{}{R\,y\,z\,w \vdash R\,y\,z\,w}\ \mathrm{AXr}}{R\,0\,x\,y, R\,y\,z\,w \vdash R\,y\,z\,w}\ \mathrm{WrL}}{R\,0\,x\,y, R\,y\,z\,w \vdash R\,x\,z\,w}\ \mathit{monr1} \quad \dfrac{\dfrac{}{z{:}B \vdash z{:}B}\ \mathrm{AXl}}{\vdots\ \mathrm{(W)}}\ z{:}B, z{:}A, R\,0\,x\,y, R\,y\,z\,w \vdash w{:}D, w{:}C, z{:}B \quad \dfrac{\dfrac{}{w{:}C \vdash w{:}C}\ \mathrm{AXl}}{\vdots\ \mathrm{(W)}}\ w{:}C, z{:}B, z{:}A, R\,0\,x\,y, R\,y\,z\,w \vdash w{:}D, w{:}C}{z{:}B, x{:}B \to C, z{:}A, R\,0\,x\,y, R\,y\,z\,w \vdash w{:}D, w{:}C}\ \to\mathrm{L} \quad .$$

The subformula property bounds the number of different formulas arising in proofs, but it does not bound how often the formulas might appear. In fact, some of the rules of $\mathrm{S(B^+)}$ repeat formulas (*monl*) or even duplicate them (the contractions). Since the latter rules are always applicable, infinite branches may arise and proof search may not terminate. To explain this problem further, let us define the *grade* of an lwff $x{:}A$ as the number of connectives in A. Then we can partition the rules of $\mathrm{S(B^+)}$ into two types:

(i) rules that 'simplify' sequents to sequents of smaller size, in the sense that we obtain lwffs of smaller grade, or we just weaken or delete formulas; and

(ii) rules that do not simplify sequents.

The contraction rules (CrL, ClL and ClR) and the monotony rules (*monl* as well as *monr1*, *monr2* and *monr3*) are rules of type (ii), while all the other rules of $\mathrm{S(B^+)}$ are of type (i).

Thus, given the subformula property, bounding proof search in $\mathrm{S(B^+)}$ reduces to the problem of bounding applications of the rules of type (ii). We solve this problem by a proof-theoretical analysis of the applications of these rules. For concreteness, we proceed in our analysis by first proving some preliminary results (§ 3.1), and then showing that we can bound applications of the monotony rules (§ 3.2) and eliminate altogether applications of the contraction rules (§ 3.3 and § 3.4).

3.1 Preliminaries

In this subsection we prove some preliminary technical results on $\mathrm{S(B^+)}$, most of which follow immediately by inspection of the rules of $\mathrm{S(B^+)}$. We begin by introducing additional notation.

We distinguish positive and negative occurrences of subformulas.

Definition 3.3 We inductively define that a subformula B of A *occurs positive* [*negative*] *in* A, in symbols $B \Subset_+ A$ [$B \Subset_- A$], as follows:

- if $B = A$, then $B \Subset_+ A$;
- if $(B \bullet C) \Subset_- A$ with $\bullet \in \{\wedge, \vee\}$, then $B \Subset_- A$ and $C \Subset_- A$;
- if $(B \bullet C) \Subset_+ A$ with $\bullet \in \{\wedge, \vee\}$, then $B \Subset_+ A$ and $C \Subset_+ A$;
- if $(B \to C) \Subset_- A$, then $B \Subset_+ A$ and $C \Subset_- A$;
- if $(B \to C) \Subset_+ A$, then $B \Subset_- A$ and $C \Subset_+ A$.

We say that $y{:}B$ occurs positive [negative] in $x{:}A$, in symbols $y{:}B \Subset_+ x{:}A$ [$y{:}B \Subset_- x{:}A$], iff B occurs positive [negative] in A.

By extension, we inductively define that an lwff $y{:}B$ *occurs positive* [*negative*] *in a multiset of lwffs* Γ, in symbols $y{:}B \Subset_+ \Gamma$ [$y{:}B \Subset_- \Gamma$], as follows:

- if $y{:}B \in \Gamma$, then $y{:}B \Subset_+ \Gamma$;
- if $y{:}B \Subset_+ x{:}A$ and $x{:}A \Subset_+ \Gamma$, then $y{:}B \Subset_+ \Gamma$;
- if $y{:}B \Subset_- x{:}A$ and $x{:}A \Subset_- \Gamma$, then $y{:}B \Subset_+ \Gamma$;
- if $y{:}B \Subset_+ x{:}A$ and $x{:}A \Subset_- \Gamma$, then $y{:}B \Subset_- \Gamma$;
- if $y{:}B \Subset_- x{:}A$ and $x{:}A \Subset_+ \Gamma$, then $y{:}B \Subset_- \Gamma$. ■

Definition 3.4 Consider a proof Π of a sequent S in $\mathrm{S(B^+)}$, and let $x{:}A$ be a particular occurrence of an lwff in S. We say that $x{:}A$ is *weak* in Π iff it is weak in every branch $\mathcal{B}$ of Π, where $x{:}A$ is weak in a branch $\mathcal{B}$ of Π iff

(i) $x{:}A$ is introduced by weakening in $\mathcal{B}$; or

(ii) $x{:}A$ is $x{:}B \wedge C$ and is introduced by an application of $\wedge$L in $\mathcal{B}$,

$$\dfrac{\begin{array}{c}\Pi_1\\ x{:}B, x{:}C, \Gamma_1, \Delta_1 \vdash \Gamma_1'\end{array}}{\begin{array}{c}x{:}B \wedge C, \Gamma_1, \Delta_1 \vdash \Gamma_1'\\ \vdots\\ x{:}B \wedge C, \Gamma, \Delta \vdash \Gamma'\end{array}}\ \wedge\mathrm{L}$$

and $x{:}B$ and $x{:}C$ are both weak in Π_1; or

(iii) $x{:}A$ is $x{:}B \wedge C$ and is introduced by an application of $\wedge$R in $\mathcal{B}$

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}B & \Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}C\end{array}}{\begin{array}{c}\Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}B \wedge C\\ \vdots\\ \Gamma, \Delta \vdash \Gamma', x{:}B \wedge C\end{array}}\ \wedge\mathrm{R}$$

and $x{:}B$ is weak in Π_1 and $x{:}C$ is weak in Π_2; or

(iv) $x{:}A$ is $x{:}B \vee C$ and is introduced by an application of $\vee$L in $\mathcal{B}$

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ x{:}B, \Gamma_1, \Delta_1 \vdash \Gamma_1' & x{:}C, \Gamma_1, \Delta_1 \vdash \Gamma_1'\end{array}}{\begin{array}{c}x{:}B \vee C, \Gamma_1, \Delta_1 \vdash \Gamma_1'\\ \vdots\\ x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'\end{array}}\ \vee\mathrm{L}$$

and $x{:}B$ is weak in Π_1 and $x{:}C$ is weak in Π_2; or

(v) $x{:}A$ is $x{:}B \vee C$ and is introduced by an application of $\vee$R in $\mathcal{B}$

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}B, x{:}C\end{array}}{\begin{array}{c}\Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}B \vee C\\ \vdots\\ \Gamma, \Delta \vdash \Gamma', x{:}B \vee C\end{array}}\ \vee\mathrm{R}$$

and $x{:}B$ and $x{:}C$ are both weak in Π_1; or

(vi) $x{:}A$ is $x{:}B \to C$ and is introduced by an application of $\to$L in $\mathcal{B}$

$$\dfrac{\begin{array}{ccc}\Pi_1 & \Pi_2 & \Pi_3\\ \Delta_1 \vdash R\,x\,y\,z & \Gamma_1, \Delta_1 \vdash \Gamma_1', y{:}B & z{:}C, \Gamma_1, \Delta_1 \vdash \Gamma_1'\end{array}}{\begin{array}{c}x{:}B \to C, \Gamma_1, \Delta_1 \vdash \Gamma_1'\\ \vdots\\ x{:}B \to C, \Gamma, \Delta \vdash \Gamma'\end{array}}\ \to\mathrm{L}$$

and $y{:}B$ is weak in Π_2 and $z{:}C$ is weak in Π_3; or

(vii) $x{:}A$ is $x{:}B \to C$ and is introduced by an application of $\to$R in $\mathcal{B}$

$$\dfrac{\begin{array}{c}\Pi_1\\ y{:}B, \Gamma_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1', z{:}C\end{array}}{\begin{array}{c}\Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}B \to C\\ \vdots\\ \Gamma, \Delta \vdash \Gamma', x{:}B \to C\end{array}}\ \to\mathrm{R}$$

and $y{:}B$ and $z{:}C$ are weak in Π_1; or

(viii) $x{:}A$ is the principal formula of an application of *monl* in $\mathcal{B}$

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ \Delta \vdash R\,0\,a\,x & \Gamma_1, \Delta_1 \vdash \Gamma_1', a{:}A\end{array}}{\begin{array}{c}\Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}A\\ \vdots\\ \Gamma, \Delta \vdash \Gamma', x{:}A\end{array}}\ \mathit{monl}$$

and $a{:}A$ is weak in Π_2; or

(ix) $x{:}A$ is the principal formula of an application of ClL in $\mathcal{B}$

$$\dfrac{\begin{array}{c}\Pi_1\\ x{:}A, x{:}A, \Gamma_1, \Delta_1 \vdash \Gamma_1'\end{array}}{\begin{array}{c}x{:}A, \Gamma_1, \Delta_1 \vdash \Gamma_1'\\ \vdots\\ x{:}A, \Gamma, \Delta \vdash \Gamma'\end{array}}\ \mathrm{ClL}$$

and both instances of $x{:}A$ are weak in Π_1; or

(x) $x{:}A$ is the principal formula of an application of ClR in $\mathcal{B}$

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}A, x{:}A\end{array}}{\begin{array}{c}\Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}A\\ \vdots\\ \Gamma, \Delta \vdash \Gamma', x{:}A\end{array}}\ \mathrm{ClR}$$

and both instances of $x{:}A$ are weak in Π_1. ■

In other words, $x{:}A$ is weak in $\mathcal{B}$ iff it is introduced by weakening in $\mathcal{B}$, or all of its subformulas of smallest grade in the proof are (in the sense that the lwffs active in the rules introducing $x{:}A$ are themselves weak in $\mathcal{B}$). If $x{:}A$ is not weak, then we sometimes say that $x{:}A$ 'leads to axioms'.

Let us call *lwff-rule* a rule that has an lwff as its principal formula. Hence, the lwff-rules of $\mathrm{S(B^+)}$ are $\wedge$L, $\wedge$R, $\vee$L, $\vee$R, $\to$L, $\to$R, WlL, WlR, ClL, ClR and *monl*. Now suppose that, reasoning forwards, we derive a sequent S from sequents $S_1, \ldots, S_n$ by first applying the lwff-rule r_1 and then applying the lwff-rule r_2, where (i) each of the premises of r_2 results from an application of r_1; (ii) all applications of r_1 have the same principal formula $x{:}A$; and (iii) the principal formula $x{:}A$ of r_1 is parametric in the application of r_2. We then say that r_1 is *permutable* over r_2, or that r_1 *permutes* over r_2, if the original inference may be replaced by one in which the sequent S is derived from $S_1, \ldots, S_n$ by applying first r_2 and then r_1. By extension, a derivation or proof Π permutes to Π' if Π' is obtained from Π by one or more permutations of lwff-rules. For example,

$$\dfrac{\dfrac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}C \quad y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}D}{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}C \wedge D}\ \wedge\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B, a{:}C \wedge D}\ \to\mathrm{R} \tag{4}$$

permutes to

$$\dfrac{\dfrac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}C}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B, a{:}C}\ \to\mathrm{R} \quad \dfrac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}D}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B, a{:}D}\ \to\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B, a{:}C \wedge D}\ \wedge\mathrm{R} \tag{5}$$

since y and z are different from a by the condition on the application of $\to$R in (4). We can also reverse this permutation, i.e. (5) permutes to (4).

Similarly,

$$\dfrac{\dfrac{a{:}C, y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B \quad a{:}D, y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{a{:}C \vee D, y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}\ \vee\mathrm{L}}{a{:}C \vee D, \Gamma, \Delta \vdash \Gamma', x{:}A \to B}\ \to\mathrm{R} \tag{6}$$

permutes to

$$\dfrac{\dfrac{a{:}D, y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{a{:}C, \Gamma, \Delta \vdash \Gamma', x{:}A \to B}\ \to\mathrm{R} \quad \dfrac{a{:}D, y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{a{:}D, \Gamma, \Delta \vdash \Gamma', x{:}A \to B}\ \to\mathrm{R}}{a{:}C \vee D, \Gamma, \Delta \vdash \Gamma', x{:}A \to B}\ \vee\mathrm{L} \tag{7}$$

since y and z are different from a by the condition on the application of $\to$R in (6). As in the previous example, we can also reverse this permutation, i.e. (7) permutes to (6).

The following lemma follows immediately by inspection of the rules; note the analogy with the sequent calculus for first-order logic, e.g. [30, Lemma 5.2.2.].

Lemma 3.5 Every lwff-rule of $\mathrm{S}(\mathrm{B}^+)$ is permutable over any other lwff-rule, except for $\to$L, which is permutable over all lwff-rules but $\to$R. ■

The reason that $\to$L is, in general, not permutable over $\to$R is that $\to$L may have the same active rwff $R\,x\,y\,z$ as $\to$R, e.g.

$$\dfrac{\dfrac{\Delta, R\,x\,y\,z \vdash R\,x\,y\,z \quad y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, y{:}C \quad z{:}D, y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{x{:}C \to D, y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}\,{\to}\mathrm{L}}{x{:}C \to D, \Gamma, \Delta \vdash \Gamma', x{:}A \to B}\,{\to}\mathrm{R} \quad .$$

More concrete examples of this situation are given later in this paper, e.g. in the derivation (46) in the proof of Theorem 3.22. There are, however, cases in which $\to$L is permutable over $\to$R, most notably the case when the active rwff of $\to$L is introduced by identity:

$$\dfrac{\dfrac{\begin{array}{c}\overline{\vdash R\,0\,a\,a}\;\mathit{iden}\\ \vdots\;\mathrm{WrL}\\ \Delta, R\,x\,y\,z \vdash R\,0\,a\,a\end{array} \quad y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}C \quad a{:}D, y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{0{:}C \to D, y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}\,{\to}\mathrm{L}}{0{:}C \to D, \Gamma, \Delta \vdash \Gamma', x{:}A \to B}\,{\to}\mathrm{R}$$

permutes to

$$\dfrac{\dfrac{\begin{array}{c}\overline{\vdash R\,0\,a\,a}\;\mathit{iden}\\ \vdots\;\mathrm{WrL}\\ \Delta \vdash R\,0\,a\,a\end{array} \quad \dfrac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B, a{:}C}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B, a{:}C}\,{\to}\mathrm{R} \quad \dfrac{a{:}D, y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{a{:}D, \Gamma, \Delta \vdash \Gamma', x{:}A \to B}\,{\to}\mathrm{R}}{0{:}C \to D, \Gamma, \Delta \vdash \Gamma', x{:}A \to B}\,{\to}\mathrm{L}$$

but only if a, y and z are distinct (by the condition on the application of $\to$R). However, as we show in Fact 3.7 below, this situation cannot appear in an $\mathrm{S}(\mathrm{B}^+)$-proof of a sequent $\vdash 0{:}Z$.

Note that $\to$R is permutable over $\to$L by introducing a WrL, e.g.

$$\dfrac{\begin{array}{c}\Pi_1\\ \Delta \vdash R\,x\,y\,z\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\Pi_2\\ b{:}A, \Gamma, \Delta, R\,a\,b\,c \vdash \Gamma', c{:}B, y{:}C\end{array}}{\Gamma, \Delta \vdash \Gamma', a{:}A \to B, y{:}C}\,{\to}\mathrm{R} \quad \dfrac{\begin{array}{c}\Pi_3\\ d{:}A, z{:}D, \Gamma, \Delta, R\,a\,d\,e \vdash \Gamma', e{:}B\end{array}}{z{:}D, \Gamma, \Delta \vdash \Gamma', a{:}A \to B}\,{\to}\mathrm{R}}{x{:}C \to D, \Gamma, \Delta \vdash \Gamma', a{:}A \to B}\,{\to}\mathrm{L}$$

permutes to

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Delta \vdash R\,x\,y\,z\end{array}}{\Delta, R\,a\,b\,c \vdash R\,x\,y\,z}\,\mathrm{WrL} \quad \dfrac{\begin{array}{c}\Pi_2\\ b{:}A, \Gamma, \Delta, R\,a\,b\,c \vdash \Gamma', c{:}B, y{:}C\end{array} \quad \begin{array}{c}\Pi_3[b/d\,, c/e]\\ b{:}A, z{:}D, \Gamma, \Delta, R\,a\,b\,c \vdash \Gamma', c{:}B\end{array}}{b{:}a, x{:}C \to D, \Gamma, \Delta, R\,a\,b\,c \vdash \Gamma', c{:}B}\,{\to}\mathrm{L}}{x{:}C \to D, \Gamma, \Delta \vdash \Gamma', a{:}A \to B}\,{\to}\mathrm{R}$$

provided that we uniformly substitute b for d and c for e in Π_3; these substitutions are allowed by the side conditions on the application of $\to$R.

One would be tempted to argue similarly that *monl* is, in general, not permutable over $\to$R since the two rule applications may have the same active rwff $R\,0\,x\,y$, e.g.

$$\dfrac{\dfrac{\Delta, R\,0\,x\,y \vdash R\,0\,x\,y \quad x{:}A, \Gamma, \Delta, R\,0\,x\,y \vdash \Gamma', x{:}B}{x{:}A, \Gamma, \Delta, R\,0\,x\,y \vdash \Gamma', y{:}B}\,\mathit{monl}}{\Gamma, \Delta \vdash \Gamma', 0{:}A \to B}\,{\to}\mathrm{R} \quad . \tag{8}$$

However, condition (iii) above is not satisfied in (8), since the lwff $y{:}B$ introduced by *monl* is active and not parametric in the application of $\to$R.

Inspection of the rules also shows that:

Fact 3.6 If the principal and active lwffs of a rule of $\mathrm{S}(\mathrm{B}^+)$ appear on different sides of $\vdash$, then they are labelled differently. ■

It then immediately follows that:

Fact 3.7 Let Π be an $\mathrm{S}(\mathrm{B}^+)$-proof of a sequent $\vdash 0{:}Z$, and consider a sequent $S = \Gamma, \Delta \vdash \Gamma'$ appearing in Π. Then no lwff of the form $0{:}A$ may appear in the antecedent of S. ■

By using Fact 3.7 it follows that we can dispose of the rule *iden* for theoremhood.[3]

Lemma 3.8 Every sequent $\vdash 0{:}Z$ provable in $\mathrm{S}(\mathrm{B}^+)$ has a proof that contains no applications of *iden*.

Proof We consider the possible applications of *iden* in an $\mathrm{S}(\mathrm{B}^+)$-proof Π of $\vdash 0{:}Z$. If Π has the form

$$\dfrac{\dfrac{\dfrac{}{\vdash R\,0\,x\,x}\ \mathit{iden}}{\vdots\ \mathrm{WrL}}}{\Delta \vdash R\,0\,x\,x} \qquad \dfrac{\begin{matrix}\Pi_1\\ \Delta \vdash R\,x\,y\,z\end{matrix}}{\begin{matrix}\Delta \vdash R\,x\,y\,z\\ \Pi_0\\ \vdash 0{:}Z\end{matrix}}\ \mathit{monr1} \qquad \text{then we simply transform it to} \qquad \begin{matrix}\Pi_1\\ \Delta \vdash R\,x\,y\,z\\ \Pi_0\\ \vdash 0{:}Z\end{matrix}\ . \tag{9}$$

We proceed analogously when *iden* provides a premise of an application of *monr2* or *monr3*. The only other case that we must consider is when *iden* provides the relational premise $R\,0\,x\,x$ of an application of $\rightarrow \mathrm{L}$. But then the principal formula of this $\rightarrow \mathrm{L}$ would be $0{:}A \rightarrow B$, which, by Fact 3.7, cannot appear on the left of $\vdash$ in a proof of $\vdash 0{:}Z$. Hence, by repeatedly performing on Π transformations similar to (9) above, we obtain an $\mathrm{S}(\mathrm{B}^+)$-proof Π' of $\vdash 0{:}Z$ that does not contain applications of *iden*. ■

Lemma 3.8 implies that:

Corollary 3.9 The sequent $\Delta \vdash R\,0\,x\,y$ is provable in an $\mathrm{S}(\mathrm{B}^+)$-proof of $\vdash 0{:}Z$ if and only if $R\,0\,x\,y \in \Delta$. ■

3.2 Bounding Monotony

We consider first the relational monotony rules, and then the rule *monl*. Another consequence of Lemma 3.8 is the eliminability of the rules *monr2* and *monr3* for theoremhood, i.e.

Lemma 3.10 Every sequent $\vdash 0{:}Z$ provable in $\mathrm{S}(\mathrm{B}^+)$ has a proof that contains no applications of *monr2* or *monr3*.

Proof Let Π be an $\mathrm{S}(\mathrm{B}^+)$-proof of $\vdash 0{:}Z$. We show how to transform Π to eliminate the highest application of *monr2* or *monr3*; the lemma follows by iterating the transformation. Without loss of generality, let such a highest application be an application of *monr2*; there are two possible forms for it.

In the first case, we have:

$$\dfrac{\dfrac{\dfrac{}{\vdash R\,0\,y\,y}\ \mathit{iden}}{\vdots\ \mathrm{WrL}}}{\Delta \vdash R\,0\,y\,y} \qquad \dfrac{\begin{matrix}\Pi_1\\ \Delta \vdash R\,x\,y\,z\end{matrix}}{\begin{matrix}\Delta \vdash R\,x\,y\,z\\ \Pi_0\\ \vdash 0{:}Z\end{matrix}}\ \mathit{monr2}\ .$$

This application of *monr2* is redundant, and it can be eliminated from Π since by Lemma 3.8 we can always transform Π so that it contains no application of *iden*.

[3] It is also possible to dispose of *iden* by building it into the special implication rules; we return to this in § 5, where we introduce the rules $\rightarrow \mathrm{L}_0$ and $\rightarrow \mathrm{R}_0$. Note however that we cannot directly eliminate *iden* when we consider *logical consequence*, i.e. proofs of sequents of the form $0{:}A \vdash 0{:}B$. The decidability of the logical consequence problem of B^+ has been established by a semantic filtration argument, and showing if and how our method provides an efficient decision procedure for this problem remains as future work.

In the second case, we must consider:

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Delta \vdash R\,0\,a\,y\end{array} \quad \begin{array}{c}\Pi_2\\ \Delta \vdash R\,x\,y\,z\end{array}}{\Delta \vdash R\,x\,a\,z}\ \mathit{monr2}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \tag{10}$$

where Π_1 and Π_2 do not contain applications of *monr2* or *monr3*. By Corollary 3.9, Π_1 follows iff $R\,0\,a\,y \in \Delta$. Thus (10) must have the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{R\,0\,a\,y \vdash R\,0\,a\,y}\ \mathrm{AXr}}{\vdots\ \mathrm{WrL}} \quad \begin{array}{c}\Pi_2\\ \Delta_1, R\,0\,a\,y \vdash R\,x\,y\,z\end{array}}{\Delta_1, R\,0\,a\,y \vdash R\,x\,a\,z}\ \mathit{monr2}}{\begin{array}{c}\Pi_3\\ a{:}A, \Gamma_2, \Delta_2, R\,0\,a\,y \vdash \Gamma_2', y{:}B\end{array}}}{\Gamma_2, \Delta_2 \vdash \Gamma_2', 0{:}A \to B}\ {\to}\mathrm{R}}{\begin{array}{c}\Pi_4\\ \vdash 0{:}Z\end{array}} \tag{11}$$

where $\Delta = \Delta_1 \cup \{R\,0\,a\,y\}$, and a and y do not occur in Γ_2, Δ_2 or Γ_2'. Consider now Π_2. It does not contain applications of *monr2* or *monr3*, and thus Π_2 is such that either $R\,x\,y\,z \in \Delta_1$, or $R\,x\,y\,z$ follows from $\Delta_1, R\,0\,a\,y$ by one or more applications of *monr1*. We show that neither of the two cases is possible.

For the first case, suppose that $R\,x\,y\,z \in \Delta_1$. Then in Π_3 or Π_4 there should be an application of ${\to}\mathrm{R}$ that has $R\,x\,y\,z$ as its active rwff, i.e.

$$\dfrac{y{:}C, \Gamma_3, \Delta_3, R\,x\,y\,z \vdash \Gamma_3', z{:}D}{\Gamma_3, \Delta_3 \vdash \Gamma_3', x{:}C \to D}\ {\to}\mathrm{R} \quad . \tag{12}$$

However, this cannot be, for otherwise the side conditions on the applications of ${\to}\mathrm{R}$ would not be satisfied. For the sake of contradiction, suppose that the application of ${\to}\mathrm{R}$ displayed in (12) occurs in Π_3. Then $R\,0\,a\,y \in \Delta_3$ since $R\,0\,a\,y$ is parametric in Π_3, and the condition on the application of the ${\to}\mathrm{R}$ in (12) is not satisfied because y occurs in two distinct rwffs. Suppose therefore, still for the sake of contradiction, that the application of ${\to}\mathrm{R}$ displayed in (12) occurs in Π_4. Then $R\,x\,y\,z \in \Delta_1$ implies $R\,x\,y\,z \in \Delta_2$, and we cannot apply the ${\to}\mathrm{R}$ that introduces $0{:}A \to B$, since its side condition, that y is an arbitrary label, is not satisfied.

For the second case, suppose that $R\,x\,y\,z$ follows from $\Delta_1, R\,0\,a\,y$ by one or more applications of *monr1*. For example, let Π_2 be

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,b \vdash R\,0\,x\,b}\ \mathrm{AXr}}{\vdots\ \mathrm{WrL}} \quad \begin{array}{c}\Pi_5\\ \Delta_3, R\,0\,x\,b, R\,0\,a\,y \vdash R\,b\,y\,z\end{array}}{\Delta_3, R\,0\,x\,b, R\,0\,a\,y \vdash R\,x\,y\,z}\ \mathit{monr1}$$

where $\Delta_1 = \Delta_3 \cup \{R\,0\,x\,b\}$ by Corollary 3.9. We can now reason as in the previous subcase: for Π_5 to exist, it must be the case that y appears in some rwff $R\,c\,y\,d$ or $R\,c\,d\,y$ in Δ_3, and this rwff is then active in an application of ${\to}\mathrm{R}$. But it is impossible to satisfy the condition on this ${\to}\mathrm{R}$, and thus such an application of *monr1* cannot appear at all in Π_2. This concludes the proof. ■

Lemma 3.10 implies that:

Lemma 3.11 Let Π be an $\mathrm{S}(\mathrm{B}^+)$-proof of the sequent $\vdash 0{:}Z$, and consider an application of *monr1* in Π

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,a \vdash R\,0\,x\,a}\ \mathrm{AXr}}{\vdots\ \mathrm{WrL}} \quad \begin{array}{c}\Pi_1\\ \Delta, R\,0\,x\,a \vdash R\,a\,y\,z\end{array}}{\Delta, R\,0\,x\,a \vdash R\,x\,y\,z}\ \mathit{monr1}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad . \tag{13}$$

Then it cannot be the case that Π_1 contains an application of *monr1* that introduces $R\,a\,y\,z$; in other words, it must be the case that $R\,a\,y\,z \in \Delta$.

Proof We can safely assume that Π does not contain applications of *iden*; if this is not the case, then we first transform Π using Lemma 3.8. Suppose, for the sake of contradiction, that $R\,a\,y\,z \notin \Delta$. Then, since we can eliminate *monr2* and *monr3* as in Lemma 3.10, and since we can always permute upwards applications of WrL (cf. Fact 3.16 below), the rwff $R\,a\,y\,z$ must be introduced by an application of *monr1*, so that (13) is

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,a \vdash R\,0\,x\,a}\ \text{AXr}}{\vdots\ \text{WrL}}}{\dfrac{\Delta_1, R\,0\,a\,b, R\,0\,x\,a \vdash R\,0\,x\,a}{\Delta_1, R\,0\,a\,b, R\,0\,x\,a \vdash R\,x\,y\,z}\ \textit{monr1}}\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{R\,0\,a\,b \vdash R\,0\,a\,b}\ \text{AXr}}{\vdots\ \text{WrL}}}{\Delta_1, R\,0\,a\,b, R\,0\,x\,a \vdash R\,0\,a\,b}\qquad \dfrac{\Pi_2}{\Delta_1, R\,0\,a\,b, R\,0\,x\,a \vdash R\,b\,y\,z}}{\Delta_1, R\,0\,a\,b, R\,0\,x\,a \vdash R\,a\,y\,z}\ \textit{monr1}$$

$$\dfrac{\Pi_0}{\vdash 0{:}Z}$$

where $\Delta = \Delta_1 \cup \{R\,0\,a\,b\}$ by Corollary 3.9. This is however not possible, since then Π_0 should contain two applications of $\to$R with active rwffs $R\,0\,a\,b$ and $R\,0\,x\,a$, but the conditions on their applications would not be satisfied as a appears in both rwffs. ■

We now focus on applications of the rule *monl*. By analogy with Lemma 3.11, we have:

Lemma 3.12 Let Π be an S(B$^+$)-proof of the sequent $\vdash 0{:}Z$. It cannot be the case that the active lwff $x{:}A$ of an application of *monl* in Π is the principal formula of another application of *monl* in Π.

Proof Let Π be an S(B$^+$)-proof of the sequent $\vdash 0{:}Z$, and suppose, for the sake of contradiction, that the active lwff $x{:}A$ of an application of *monl* in Π is introduced by another application of *monl*. Then, by Lemma 3.8 and by the permutability of the rules, Π must have the form

$$\dfrac{\dfrac{\dfrac{\dfrac{}{R\,0\,x\,y \vdash R\,0\,x\,y}\ \text{AXr}}{\vdots\ \text{WrL}}}{\Delta, R\,0\,a\,x, R\,0\,x\,y \vdash R\,0\,x\,y}\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{R\,0\,a\,x \vdash R\,0\,a\,x}\ \text{AXr}}{\vdots\ \text{WrL}}}{\Delta, R\,0\,a\,x, R\,0\,x\,y \vdash R\,0\,a\,x}\qquad \dfrac{\Pi_2}{\Gamma, \Delta, R\,0\,a\,x, R\,0\,x\,y \vdash \Gamma', a{:}A}}{\Gamma, \Delta, R\,0\,a\,x, R\,0\,x\,y \vdash \Gamma', x{:}A}\ \textit{monl}}{\Gamma, \Delta, R\,0\,a\,x, R\,0\,x\,y \vdash \Gamma', y{:}A}\ \textit{monl}$$

$$\dfrac{\Pi_0}{\vdash 0{:}Z}\ .$$

This is however not possible, since then Π_0 should contain two applications of $\to$R with active rwffs $R\,0\,a\,x$ and $R\,0\,x\,y$, but the conditions on their applications would not be satisfied as x appears in both rwffs. ■

We now show that we can restrict applications of the rule *monl* to atomic formulas, i.e. to lwffs of grade 0. Note that this is the inverse of what is usually done when presenting the semantics of relevance logics, where a monotony property is first postulated for atomic formulas, and then generalized to arbitrary formulas by induction on their structure, cf. § 2 and, e.g., [2, 29].

Lemma 3.13 Every sequent provable in S(B$^+$) has a proof in which all the applications of *monl* have active formula $x{:}A$ of grade 0, i.e. A is atomic.

Proof Consider an application of *monl*

$$\dfrac{\dfrac{\Pi_0}{\Delta \vdash R\,0\,x\,y}\qquad \dfrac{\Pi_1}{\Gamma, \Delta \vdash \Gamma', x{:}A}}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ \textit{monl} \tag{14}$$

where A is not atomic. We proceed by induction on the structure of the proof Π_1, where, by the permutability of the rules, we can assume that the lowest rule application in Π_1 introduces the lwff $x{:}A$. If the lowest rule in Π_1 is an application of WlR, then we conclude by replacing this weakening and the application of *monl* with a weakening directly introducing $y{:}A$ and applying the induction hypothesis.

If the lowest rule in Π_1 is an application of $\wedge$R, then A is $B \wedge C$ and (14) is

$$\dfrac{\dfrac{\Pi_0}{\Delta \vdash R\,0\,x\,y} \quad \dfrac{\dfrac{\Pi_2}{\Gamma, \Delta \vdash \Gamma', x{:}B} \quad \dfrac{\Pi_3}{\Gamma, \Delta \vdash \Gamma', x{:}C}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C}\,\wedge\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', y{:}B \wedge C}\,monl$$

which we can transform to

$$\dfrac{\dfrac{\dfrac{\Pi_0}{\Delta \vdash R\,0\,x\,y} \quad \dfrac{\Pi_2}{\Gamma, \Delta \vdash \Gamma', x{:}B}}{\Gamma, \Delta \vdash \Gamma', y{:}B}\,monl \quad \dfrac{\dfrac{\Pi_0}{\Delta \vdash R\,0\,x\,y} \quad \dfrac{\Pi_3}{\Gamma, \Delta \vdash \Gamma', x{:}C}}{\Gamma, \Delta \vdash \Gamma', y{:}C}\,monl}{\Gamma, \Delta \vdash \Gamma', y{:}B \wedge C}\,\wedge\mathrm{R}\quad .$$

We conclude by applying the induction hypotheses.

If the lowest rule in Π_1 is an application of $\vee$R, then A is $B \vee C$ and (14) is

$$\dfrac{\dfrac{\Pi_0}{\Delta \vdash R\,0\,x\,y} \quad \dfrac{\dfrac{\Pi_2}{\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}C}}{\Gamma, \Delta \vdash \Gamma', x{:}B \vee C}\,\vee\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', y{:}B \vee C}\,monl$$

which we can transform to

$$\dfrac{\dfrac{\Pi_0}{\Delta \vdash R\,0\,x\,y} \quad \dfrac{\dfrac{\Pi_0}{\Delta \vdash R\,0\,x\,y} \quad \dfrac{\Pi_2}{\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}C}}{\Gamma, \Delta \vdash \Gamma', x{:}B, y{:}C}\,monl}{\dfrac{\Gamma, \Delta \vdash \Gamma', y{:}B, y{:}C}{\Gamma, \Delta \vdash \Gamma', y{:}B \vee C}\,\vee\mathrm{R}}\,monl\quad .$$

We conclude by applying the induction hypotheses.

If the lowest rule in Π_1 is an application of $\to$R, then A is $B \to C$ and (14) is

$$\dfrac{\dfrac{\Pi_0}{\Delta \vdash R\,0\,x\,y} \quad \dfrac{\dfrac{\Pi_2}{u{:}B, \Gamma, \Delta, R\,x\,u\,v \vdash \Gamma', v{:}C}}{\Gamma, \Delta \vdash \Gamma', x{:}B \to C}\,\to\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', y{:}B \to C}\,monl$$

which we can transform to

$$\dfrac{\dfrac{\Pi_2'}{u{:}B, \Gamma, \Delta, R\,y\,u\,v \vdash \Gamma', v{:}C}}{\Gamma, \Delta \vdash \Gamma', y{:}B \to C}\,\to\mathrm{R}$$

where Π_2' is obtained from Π_2 by substituting $R\,y\,u\,v$ for $R\,x\,u\,v$, and by appropriately transforming all inferences in Π_2 in which $R\,x\,u\,v$ is active.[4] For an example, let $\Gamma = \Gamma_1 \cup \{x{:}D \to E\}$ and let Π_2 contain the derivation

$$\dfrac{\dfrac{\Pi_3}{\Delta, R\,x\,u\,v \vdash R\,x\,u\,v} \quad \dfrac{\Pi_4}{u{:}B, \Gamma_1, \Delta, R\,x\,u\,v \vdash \Gamma', v{:}C, u{:}D} \quad \dfrac{\Pi_5}{v{:}E, u{:}B, \Gamma_1, \Delta, R\,x\,u\,v \vdash \Gamma', v{:}C}}{u{:}B, \Gamma_1, x{:}D \to E, \Delta, R\,x\,u\,v \vdash \Gamma', v{:}C}\,\to\mathrm{L}\quad .$$

[4] Note that the substitution of $R\,y\,u\,v$ for $R\,x\,u\,v$ is allowed since $R\,x\,u\,v$ must be introduced by WrL. This is essentially equivalent to performing a *cut* on rwffs:

$$\dfrac{\dfrac{\dfrac{\Pi_0}{\Delta \vdash R\,0\,x\,y} \quad \dfrac{\dfrac{\dfrac{}{R\,y\,u\,v \vdash R\,y\,u\,v}\,\mathrm{AXr}}{\vdots}\,\mathrm{WrL}}{\Delta, R\,y\,u\,v \vdash R\,x\,u\,v}\,monr1 \quad \dfrac{\Pi_2}{u{:}B, \Gamma, \Delta, R\,x\,u\,v \vdash \Gamma', v{:}C}}{u{:}B, \Gamma, \Delta, R\,y\,u\,v \vdash \Gamma', v{:}C}\,cut}{\Gamma, \Delta \vdash \Gamma', y{:}B \to C}\,\to\mathrm{R}\quad .$$

We transform this to

$$\dfrac{\overset{\Pi_6}{\Delta, R\,y\,u\,v \vdash R\,x\,u\,v} \quad \overset{\Pi_4'}{u{:}B, \Gamma_1, \Delta, R\,y\,u\,v \vdash \Gamma', v{:}C, u{:}D} \quad \overset{\Pi_5'}{v{:}E, u{:}B, \Gamma_1, \Delta, R\,y\,u\,v \vdash \Gamma', v{:}C}}{u{:}B, \Gamma_1, x{:}D \to E, \Delta, R\,y\,u\,v \vdash \Gamma', v{:}C} \to \mathrm{L}$$

where Π_6 is

$$\dfrac{\dfrac{\overset{\Pi_0}{\Delta \vdash R\,0\,x\,y}}{\Delta, R\,y\,u\,v \vdash R\,0\,x\,y}\ \mathrm{WrL} \qquad \dfrac{\dfrac{}{R\,y\,u\,v \vdash R\,y\,u\,v}\ \mathrm{AXr}}{\Delta, R\,y\,u\,v \vdash R\,y\,u\,v}\ \mathrm{WrL}}{\Delta, R\,y\,u\,v \vdash R\,x\,u\,v}\ \mathit{monr1} \quad .$$

That is, we 'push' monotony from lwffs to rwffs.

If the lowest rule in Π_1 is an application of *monl*, then (14) is

$$\dfrac{\overset{\Pi_0}{\Delta \vdash R\,0\,x\,y} \qquad \dfrac{\overset{\Pi_2}{\Delta \vdash R\,0\,a\,x} \quad \overset{\Pi_3}{\Gamma, \Delta \vdash \Gamma', a{:}A}}{\Gamma, \Delta \vdash \Gamma', x{:}A}\ \mathit{monl}}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ \mathit{monl}$$

which we can transform to

$$\dfrac{\dfrac{\overset{\Pi_2}{\Delta \vdash R\,0\,a\,x} \quad \overset{\Pi_0}{\Delta \vdash R\,0\,x\,y}}{\Delta \vdash R\,0\,a\,y}\ \mathit{monr1} \qquad \overset{\Pi_3}{\Gamma, \Delta \vdash \Gamma', a{:}A}}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ \mathit{monl} \quad .$$

We conclude by applying the induction hypothesis. ■

The case for $\to$R in the proof of Lemma 3.13 can alternatively be justified by using the following lemma.

Lemma 3.14 The rule *reverse monotony (for lwffs)*

$$\dfrac{\Delta \vdash R\,0\,x\,y \quad y{:}A, \Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\ \mathit{rev\text{-}monl}$$

is derivable in S(B$^+$).

Proof We consider an application of *rev-monl*

$$\dfrac{\overset{\Pi_0}{\Delta \vdash R\,0\,x\,y} \quad \overset{\Pi_1}{y{:}A, \Gamma, \Delta \vdash \Gamma'}}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\ \mathit{rev\text{-}monl} \tag{15}$$

and proceed by induction on the structure of Π_1. If the lowest rule applied in Π_1 has principal formula different from $y{:}A$, we conclude by the induction hypothesis. Therefore suppose now that the $y{:}A$ is the principal formula of the lowest rule in Π_1; we proceed by induction on A.

If A is $B \wedge C$, then (15) has the form

$$\dfrac{\overset{\Pi_0}{\Delta \vdash R\,0\,x\,y} \quad \dfrac{\overset{\Pi_2}{y{:}B, y{:}C, \Gamma, \Delta \vdash \Gamma'}}{y{:}B \wedge C, \Gamma, \Delta \vdash \Gamma'}\ \wedge\mathrm{L}}{x{:}B \wedge C, \Gamma, \Delta \vdash \Gamma'}\ \mathit{rev\text{-}monl}$$

and we conclude by the induction hypothesis.

If A is $B \vee C$, then we also conclude straightforwardly by applying the induction hypothesis.

If A is $B \to C$, then (15) has the form

$$\dfrac{\Pi_0 \quad \dfrac{\Pi_2 \quad \Pi_3 \quad \Pi_4}{\Delta \vdash R\,y\,u\,v \quad \Gamma, \Delta \vdash \Gamma', u{:}B \quad v{:}C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{\Delta \vdash R\,0\,x\,y \qquad y{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\textit{rev-monl}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}$$

which we can transform to

$$\dfrac{\dfrac{\Pi_0 \quad \Pi_2}{\Delta \vdash R\,0\,x\,y \quad \Delta \vdash R\,y\,u\,v}\,\textit{monr1} \qquad \Pi_3 \quad \Pi_4}{\Delta \vdash R\,x\,u\,v \qquad\qquad \Gamma, \Delta \vdash \Gamma', u{:}B \quad v{:}C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,.$$

This concludes the proof of the lemma. ■

Given this result, we can then justify the case for $\to$R in the proof of Lemma 3.13 by transforming all inferences in Π_2 in which $R\,x\,u\,v$ is active by appropriately applying *rev-monl*. For example, we transform

$$\dfrac{\Pi_0 \qquad \dfrac{\dfrac{\dfrac{}{R\,x\,u\,v \vdash R\,x\,u\,v}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta, R\,x\,u\,v \vdash R\,x\,u\,v} \qquad \dfrac{\Pi_4 \qquad\qquad \Pi_5}{u{:}B, \Gamma_1, \Delta, R\,x\,u\,v \vdash \Gamma', v{:}C, u{:}D \quad v{:}E, u{:}B, \Gamma_1, \Delta, R\,x\,u\,v \vdash \Gamma', v{:}C}\,{\to}\mathrm{L}}{u{:}B, \Gamma_1, x{:}D \to E, \Delta, R\,x\,u\,v \vdash \Gamma', v{:}C}\,{\to}\mathrm{R}}{\Gamma_1, x{:}D \to E, \Delta \vdash x{:}B \to C}\,\textit{monl}}{\Delta \vdash R\,0\,x\,y \qquad\qquad \Gamma_1, x{:}D \to E, \Delta \vdash y{:}B \to C}$$

to

$$\dfrac{\dfrac{\Pi_0}{\Delta \vdash R\,0\,x\,y}\,\mathrm{WrL}}{\Delta, R\,y\,u\,v \vdash R\,0\,x\,y} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{R\,y\,u\,v \vdash R\,y\,u\,v}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta, R\,y\,u\,v \vdash R\,y\,u\,v} \qquad \dfrac{\Pi_4 \qquad\qquad \Pi_5}{u{:}B, \Gamma_1, \Delta, R\,y\,u\,v \vdash \Gamma', v{:}C, u{:}D \quad v{:}E, u{:}B, \Gamma_1, \Delta, R\,y\,u\,v \vdash \Gamma', v{:}C}\,{\to}\mathrm{L}}{u{:}B, \Gamma_1, y{:}D \to E, \Delta, R\,y\,u\,v \vdash \Gamma', v{:}C}\,\textit{rev-monl}}{u{:}B, \Gamma_1, x{:}D \to E, \Delta, R\,y\,u\,v \vdash \Gamma', v{:}C}\,{\to}\mathrm{R}}{\Gamma_1, x{:}D \to E, \Delta \vdash y{:}B \to C}\,.$$

We can obtain an equivalent system by restricting also the axiom AXl to atomic lwffs.

Lemma 3.15 Every sequent provable in S(B$^+$) has a proof in which all the axioms employed contain no connectives, i.e. the formula A in each application of AXl is atomic. ■

For example, we replace

$$\dfrac{}{x{:}A \to B \vdash x{:}A \to B}\,\mathrm{AXl}$$

with

$$\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr} \qquad \dfrac{\dfrac{}{y{:}A \vdash y{:}A}\,\mathrm{AXl}}{\vdots\,(\mathrm{W})} \qquad \dfrac{\dfrac{}{z{:}B \vdash z{:}B}\,\mathrm{AXl}}{\vdots\,(\mathrm{W})}}{y{:}A, R\,x\,y\,z \vdash z{:}B, y{:}A \qquad z{:}B, y{:}A, R\,x\,y\,z \vdash z{:}B}\,{\to}\mathrm{L}}{x{:}A \to B, y{:}A, R\,x\,y\,z \vdash z{:}B}\,{\to}\mathrm{R}}{x{:}A \to B \vdash x{:}A \to B}\,.$$

In the remaining part of the paper we will thus restrict applications of AXl and *monl* to atomic lwffs. That is, we consider the restricted sequent system in Figure 3, which, for simplicity, we also call S(B$^+$). Note that we could consider also the other side of the spectrum and dispense with the weakening rules by building them into the axioms and the rule *iden*, i.e.

$$\dfrac{}{x{:}A, \Gamma, \Delta \vdash \Gamma', x{:}A}\,, \qquad \dfrac{}{\Delta, R\,x\,y\,z \vdash R\,x\,y\,z} \quad \text{and} \quad \dfrac{}{\Delta \vdash R\,0\,x\,x}\,.$$

This follows because, since all rules permute over WrL, WlL and WlR, we have:

Axioms:

$$\frac{}{x{:}A \vdash x{:}A}\ \mathrm{AXl}\ (\text{where } A \text{ is atomic}), \qquad \frac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\ \mathrm{AXr}.$$

Logical rules:

$$\frac{x{:}A, x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \wedge B, \Gamma, \Delta \vdash \Gamma'}\ \wedge\mathrm{L}, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A \quad \Gamma, \Delta \vdash \Gamma', x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \wedge B}\ \wedge\mathrm{R},$$

$$\frac{x{:}A, \Gamma, \Delta \vdash \Gamma' \quad x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \vee B, \Gamma, \Delta \vdash \Gamma'}\ \vee\mathrm{L}, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \vee B}\ \vee\mathrm{R},$$

$$\frac{\Delta \vdash R\,x\,y\,z \quad \Gamma, \Delta \vdash \Gamma', y{:}A \quad z{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \to B, \Gamma, \Delta \vdash \Gamma'}\ \to\mathrm{L}, \qquad \frac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B}\ \to\mathrm{R}.$$

In $\to$R, y and z are distinct and do not occur in $\Gamma, \Delta \vdash \Gamma', x{:}A \to B$.

Structural rules:

$$\frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta, R\,x\,y\,z \vdash \Gamma'}\ \mathrm{WrL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\ \mathrm{WlL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma', x{:}A}\ \mathrm{WlR},$$

$$\frac{\Delta, R\,a\,b\,c, R\,a\,b\,c \vdash R\,x\,y\,z}{\Delta, R\,a\,b\,c \vdash R\,x\,y\,z}\ \mathrm{CrL}, \qquad \frac{x{:}A, x{:}A, \Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\ \mathrm{ClL}, \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}A}{\Gamma, \Delta \vdash \Gamma', x{:}A}\ \mathrm{ClR}.$$

Monotony for lwffs:

$$\frac{\Delta \vdash R\,0\,x\,y \quad \Gamma, \Delta \vdash \Gamma', x{:}A}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ \mathit{monl}\ (\text{where } A \text{ is atomic and } R\,0\,x\,y \in \Delta).$$

Relational monotony rule:

$$\frac{\Delta \vdash R\,0\,a\,x \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,a\,y\,z}\ \mathit{monr1}\ (\text{where } R\,0\,a\,x \in \Delta \text{ and } R\,x\,y\,z \in \Delta).$$

Figure 3: The restricted sequent system S(B^+)

Fact 3.16 Each S(B$^+$)-proof can be transformed so that all weakenings in the proof occur immediately below the axioms. ■

This also tells us that weakening does not constitute a problem in establishing the space complexity of provability in S(B$^+$). Contractions play instead a crucial role in complexity analysis and we will now focus on these rules, showing that we can eliminate all their applications in proofs of theorems of S(B$^+$).

3.3 Eliminating Contraction

As a first result in the analysis of the contraction rules, we observe that:

Lemma 3.17 The rule CrL is eliminable in S(B$^+$). ■

This follows because contracted rwffs can only be introduced in the antecedent of a sequent by applications of the rule WrL. Therefore, we just need to delete both the CrL and the corresponding WrL, e.g. we transform

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Delta_1 \vdash R\,x\,y\,z\end{array}}{\Delta_1, R\,a\,b\,c \vdash R\,x\,y\,z}\ \text{WrL}}{\dfrac{\begin{array}{c}\Pi_2\\ \Delta_2, R\,a\,b\,c, R\,a\,b\,c \vdash R\,x\,y\,z\end{array}}{\Delta_2, R\,a\,b\,c \vdash R\,x\,y\,z}\ \text{CrL}} \quad \text{to} \quad \begin{array}{c}\Pi_1\\ \Delta_1 \vdash R\,x\,y\,z\\ \Pi_2^\dagger\\ \Delta_2, R\,a\,b\,c \vdash R\,x\,y\,z\end{array}$$

where, here and in the following, we use '$\dagger$' to denote that in the derivation $\Pi_i^\dagger$ we apply the same sequence of rules applied in the derivation Π_i, with the same active and principal formulas; in other words, Π_i and $\Pi_i^\dagger$ differ only in their parametric formulas.

As a preliminary for the analysis of ClR, we now introduce a 'relevant' analogue of the intuitionistic disjunction property. One of the consequences of cut-elimination in sequent systems for propositional intuitionistic logic is the *disjunction property*, if $\vdash A \vee B$ is provable then so is $\vdash A$ or $\vdash B$, which reflects the constructive character of the logic [30, 34]. The property holds also under hypotheses,

$$\text{if } \Gamma \vdash A \vee B \text{ is provable then so is } \Gamma \vdash A \text{ or } \Gamma \vdash B\,,$$

provided that the hypotheses in Γ are *Harrop formulas* [34], where the class $\mathcal{H}$ of Harrop formulas is inductively defined by: (i) $p \in \mathcal{H}$ for p a propositional variable, (ii) $A \wedge B \in \mathcal{H}$ if $A \in \mathcal{H}$ and $B \in \mathcal{H}$, and (iii) $A \supset B \in \mathcal{H}$ if $B \in \mathcal{H}$, where $\supset$ is intuitionistic implication.

Since $C \vee D \notin \mathcal{H}$, the restriction to Harrop formulas ensures that there is no disjunctive formula $C \vee D$ that occurs positive in Γ, i.e. $C \vee D \not\in_+ \Gamma$. This amounts to ensuring that $\Gamma \vdash A \vee B$ is not the conclusion of an application of the branching intuitionistic left disjunction rule ($\vee$L), e.g.

$$\dfrac{C, \Gamma_1 \vdash A \vee B \quad D, \Gamma_1 \vdash A \vee B}{C \vee D, \Gamma_1 \vdash A \vee B}\ (\vee\text{L})$$

where $\Gamma = \{C \vee D\} \cup \Gamma_1$. For example, the property fails for $\Gamma = \{B \vee A\}$, since we can prove $B \vee A \vdash A \vee B$ but the proof requires branching so that neither $B \vee A \vdash A$ nor $B \vee A \vdash B$ are provable.

Note that ($\vee$L) is the only rule that gives rise to 'dangerous' (for the satisfiability of the disjunction property) branchings in unlabelled sequent systems for intuitionistic logic, since the intuitionistic left implication rule (with built-in contraction) has the form

$$\dfrac{C \supset D, \Gamma \vdash C \quad D, \Gamma \vdash E}{C \supset D, \Gamma \vdash E}\ (\supset\text{L})$$

and it therefore suffices to require that $D \in \mathcal{H}$.

A property analogous to the intuitionistic disjunction property holds for $\rightarrow$ in S(B$^+$), i.e. we have:

$$\text{if } \vdash x{:}A_1 \rightarrow B_1, x{:}A_2 \rightarrow B_2 \text{ is provable, then so is } \vdash x{:}A_1 \rightarrow B_1 \text{ or } \vdash x{:}A_2 \rightarrow B_2\,.$$

(This can be easily checked via a semantic argument.) To generalize this property to hold under hypotheses, as we do in Proposition 3.19 below, we first define (Definition 3.18) an analogue of the branchings caused

by formulas that are not Harrop: in $\mathrm{S(B^+)}$ we need to consider only the branchings caused by applications of the left disjunction rule with principal formula labelled with x or with a world 'preceding' x, and the branchings caused by applications of the left implication rule with principal formula labelled with a world 'preceding' x.[5]

Definition 3.18 The *class* $\mathfrak{B}(x,\Delta)$ *of* x*-branching lwffs with respect to a multiset of rwffs* Δ *in* $\mathrm{S(B^+)}$ is inductively defined by

(i) $a{:}A \vee B \in \mathfrak{B}(x,\Delta)$ if either $a = x$ or there exists a label b such that $\Delta \vdash R\,a\,b\,x$ is provable,

(ii) $a{:}A \to B \in \mathfrak{B}(x,\Delta)$ if there exist labels b and c such that $\Delta \vdash R\,a\,b\,c$ is provable in $\mathrm{S(B^+)}$ and $c{:}B \in \mathfrak{B}(x,\Delta)$.

By extension, given a sequent $\Gamma, \Delta \vdash \Gamma'$

Γ is x*-branching with respect to* Δ *in* $\mathrm{S(B^+)}$ if $a{:}A \in \mathfrak{B}(x,\Delta)$ for some $a{:}A \Subset_+ \Gamma$. ■

Proposition 3.19 [$\to$*-disjunction property*] Consider a $\mathrm{S(B^+)}$-proof Π of the sequent

$$\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, x{:}A_2 \to B_2, \ldots, x{:}A_n \to B_n$$

where

$$\Gamma \text{ is not } x\text{-branching with respect to } \Delta \text{ in } \mathrm{S(B^+)}\,. \tag{16}$$

Then for some i, with $1 \leq i \leq n$, there exists an $\mathrm{S(B^+)}$-proof of

$$\Gamma, \Delta \vdash \Gamma', x{:}A_i \to B_i\,.$$

Proof We prove the proposition for $n = 2$; the extension to the general case is straightforward. Let the sequent

$$\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, x{:}A_2 \to B_2$$

satisfy condition (16), and let Π be its $\mathrm{S(B^+)}$-proof.

Observe that by the permutability of the rules we can assume that Π does not contain any contraction of $x{:}A_1 \to B_1$ or $x{:}A_2 \to B_2$. If this is not the case, e.g. if the lowest rule application in Π is a contraction of $x{:}A_1 \to B_1$,

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, x{:}A_1 \to B_1, x{:}A_2 \to B_2\end{array}}{\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ \mathrm{ClR} \quad ,$$

then we consider the proof Π_1 of the sequent $\Gamma, \Delta \vdash \Gamma'_1, x{:}A_1 \to B_1, x{:}A_2 \to B_2$ with $\Gamma'_1 = \Gamma' \cup \{x{:}A_1 \to B_1\}$ and show that there is a proof of $\Gamma, \Delta \vdash \Gamma'_1, x{:}A_i \to B_i$. Similarly, if one of the two lwffs $x{:}A_i \to B_i$ is contracted somewhere in Π_1, then we simply permute this ClR downwards over the rules following it.

We now proceed by induction on the structure of Π. Since the sequent cannot be an axiom, we distinguish two cases, depending on the lowest rule in Π.

[**case** 1] If the rule has principal formula other than $x{:}A_i \to B_i$, then, since Γ is not x-branching with respect to Δ in $\mathrm{S(B^+)}$, we conclude by the induction hypothesis.

[**case** 2] If the rule has principal formula $x{:}A_i \to B_i$, then it is either WlR or $\to$R. (Recall that by Lemma 3.13 we only consider applications of *monl* which introduce atomic lwffs.)

[5] Note that choosing between $x{:}A_1 \to B_1$ and $x{:}A_2 \to B_2$ constitutes a backtracking point in a backwards proof: if the chosen lwff does not allow us to close the proof, we must backtrack and try the other lwff.

[**case** 2.1] If the rule is an application of WlR with principal formula $x{:}A_i \to B_i$, then we conclude trivially; for example, if Π is

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1\end{array}}{\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ \mathrm{WlR}$$

then Π_1 is the desired proof.

[**case** 2.2] Suppose that the rule is an application of $\to$R with principal formula $x{:}A_1 \to B_1$. Then Π has the form

$$\dfrac{\begin{array}{c}\Pi_1\\ y{:}A_1, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B_1, x{:}A_2 \to B_2\end{array}}{\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ \to\mathrm{R} \tag{17}$$

and we distinguish two cases, depending on how $x{:}A_2 \to B_2$ is introduced in Π_1.

[**case** 2.2.1] If $x{:}A_2 \to B_2$ is the principal formula of an application of WlR, then (17) has the form:

$$\dfrac{\begin{array}{c}\Pi_3\\ \dfrac{\begin{array}{c}\Pi_2\\ \Gamma_1, \Delta_1 \vdash \Gamma_1'\end{array}}{\Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}A_2 \to B_2}\ \mathrm{WlR}\\ y{:}A_1, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B_1, x{:}A_2 \to B_2\end{array}}{\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ \to\mathrm{R} \quad . \tag{18}$$

Since $x{:}A_2 \to B_2$ is a parametric formula in Π_3, we can delete the application of WlR that introduces it and transform (18) to conclude as follows:

$$\dfrac{\begin{array}{c}\Pi_2\\ \Gamma_1, \Delta_1 \vdash \Gamma_1'\\ \Pi_3^{\dagger}\\ y{:}A_1, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B_1\end{array}}{\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1}\ \to\mathrm{R} \quad .$$

[**case** 2.2.2] If $x{:}A_2 \to B_2$ is the principal formula of an application of $\to$R, then (17) has the form:

$$\dfrac{\begin{array}{c}\Pi_3\\ \dfrac{\begin{array}{c}\Pi_2\\ u{:}A_2, \Gamma_1, \Delta_1, R\,x\,u\,v \vdash \Gamma_1', v{:}B_2\end{array}}{\Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}A_2 \to B_2}\ \to\mathrm{R}\\ y{:}A_1, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B_1, x{:}A_2 \to B_2\end{array}}{\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ \to\mathrm{R} \quad . \tag{19}$$

Since $x{:}A_2 \to B_2$ is a parametric formula in Π_3, we can postpone its introduction, i.e., by Lemma 3.5, we permute the uppermost $\to$R over the rules below it and transform (19) to:

$$\dfrac{\begin{array}{c}\Pi_2\\ u{:}A_2, \Gamma_1, \Delta_1, R\,x\,u\,v \vdash \Gamma_1', v{:}B_2\\ \Pi_3^{\dagger}\\ \dfrac{y{:}A_1, u{:}A_2, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1, v{:}B_2}{y{:}A_1, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B_1, x{:}A_2 \to B_2}\ \to\mathrm{R}\end{array}}{\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ \to\mathrm{R} \quad . \tag{20}$$

During the transformation it might be necessary to rename some labels to avoid clashes, for example, when y, z, u and v are not distinct, or when the first rule in Π_3 is a weakening that introduces a formula containing the labels u or v.

Now we have a proof

$$\Pi_4 = \left\{ \begin{array}{c} \Pi_2 \\ u{:}A_2, \Gamma_1, \Delta_1, R\,x\,u\,v \vdash \Gamma_1', v{:}B_2 \\ \Pi_3^\dagger \end{array} \right.$$

of

$$y{:}A_1, u{:}A_2, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1, v{:}B_2\,.$$

We show that then there is a proof of either

$$y{:}A_1, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B_1 \qquad \text{or} \qquad u{:}A_2, \Gamma, \Delta, R\,x\,u\,v \vdash \Gamma', v{:}B_2\,.$$

First, note that $R\,x\,y\,z$ and $R\,x\,u\,v$ must both be introduced by applications of WrL in Π_4. Most importantly, since y and z, and u and v are arbitrary worlds pairwise related to x in $\mathrm{S}(\mathrm{B}^+)$, it follows that y, z, u and v are all distinct and are not interrelated by the compossibility relation in any way. (For example, there are no rwffs that relate y, z and u, e.g. $R\,y\,z\,u$.) This implies that $y{:}A_1$ and $u{:}A_2$, and $z{:}B_1$ and $v{:}B_2$ are pairwise independent, so that at least one lwff of each pair is weak in Π_4. In other words, one of $y{:}A_1$ and $u{:}A_2$ is introduced by weakening in Π_4, or so are the formulas it is inferred from; similarly for $z{:}B_1$ and $v{:}B_2$. By deleting these weakenings we obtain a proof Π_4' of either

$$y{:}A_1, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1 \tag{21}$$

or

$$u{:}A_2, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', v{:}B_2\,. \tag{22}$$

This is, however, not enough to conclude, since in both cases we must dispose of the additional rwff in the antecedent. Suppose that we have:

$$\begin{array}{c} \Pi_4' \\ y{:}A_1, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1 \end{array}\,. \tag{23}$$

We dispose of $R\,x\,u\,v$ as follows; if Π_4' is a proof of (22) we proceed analogously. If the application of WrL that introduces $R\,x\,u\,v$ is the lowest step in Π_4', then (23) has the form

$$\dfrac{\begin{array}{c}\Pi_5 \\ y{:}A_1, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B_1\end{array}}{y{:}A_1, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1}\ \mathrm{WrL}$$

and Π_5 is the desired proof.

If the WrL that introduces $R\,x\,u\,v$ is not the lowest step in Π_4', then we can delete it provided that we modify the steps where $R\,x\,u\,v$ is active. $R\,x\,u\,v$ can only be active in applications of $\to\mathrm{L}$ with principal formula of the form $x{:}C \to D$, e.g. when (23) has the form

$$\begin{array}{c} \Pi_7 \qquad\qquad \Pi_8 \qquad\qquad \Pi_9 \\ \dfrac{\Delta_2, R\,x\,u\,v \vdash R\,x\,u\,v \quad \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2', u{:}C \quad v{:}D, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}{x{:}C \to D, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}\ {\to}\mathrm{L} \\ \Pi_6 \\ y{:}A_1, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1 \end{array} \tag{24}$$

or in applications of relational rules to introduce $a{:}E \to F$ by $\to\mathrm{L}$ for some a such that 0 and a are compossible with respect to x, e.g. when (23) has the form

$$\begin{array}{c} \Pi_7 \qquad\quad \Pi_8 \\ \dfrac{\Delta_2, R\,x\,u\,v \vdash R\,0\,a\,x \quad \Delta_2, R\,x\,u\,v \vdash R\,x\,u\,v}{\Delta_2, R\,x\,u\,v \vdash R\,a\,u\,v}\ \mathit{monr1} \qquad \Pi_9 \qquad\qquad \Pi_{10} \\ \dfrac{\qquad\qquad\qquad\qquad\qquad\qquad \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2', u{:}E \quad v{:}F, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}{a{:}E \to F, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}\ {\to}\mathrm{L} \\ \Pi_6 \\ y{:}A_1, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1 \end{array}\,. \tag{25}$$

We replace these applications of $\to$L by suitable steps, which depend on how the subformulas of $x{:}C \to D$ or $a{:}E \to F$ are introduced. For example, suppose that (24) has the form

$$\dfrac{\dfrac{\Pi_7}{\Delta_2, R\,x\,u\,v \vdash R\,x\,u\,v} \quad \dfrac{\dfrac{\dfrac{\Pi_{10}}{\Gamma_1, \Delta_1 \vdash \Gamma_1'}}{\Gamma_1, \Delta_1, R\,x\,u\,v \vdash \Gamma_1'}\,\mathrm{WrL} \quad \dfrac{\dfrac{\Pi_{11}}{\Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}}{\Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2', u{:}C}\,\mathrm{WlR}}{x{:}C \to D, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}\,{\to}\mathrm{L} \quad \dfrac{\dfrac{\dfrac{\Pi_{12}}{\Gamma_3, \Delta_3 \vdash \Gamma_3'}}{\Gamma_3, \Delta_3, R\,x\,u\,v \vdash \Gamma_3'}\,\mathrm{WrL} \quad \dfrac{\Pi_{13}}{v{:}D, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}}{x{:}C \to D, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}\,{\to}\mathrm{L}}}{y{:}A_1, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1}\,\Pi_6 \tag{26}$$

where $R\,x\,u\,v$ is not active in Π_6 and Π_{11}. In this case, after having deleted the WrL that introduces $R\,x\,u\,v$ in the middle branch, we also delete Π_7 and the application of $\to$L, and 'invert' (from the right to the left of $\vdash$) and 'blow up' the weakening: we apply WlL to introduce $x{:}C \to D$ directly and transform (26) to

$$\dfrac{\dfrac{\dfrac{\Pi_{10}}{\Gamma_1, \Delta_1 \vdash \Gamma_1'}}{\Gamma_1, \Delta_1, R\,x\,u\,v \vdash \Gamma_1'}\,\mathrm{WrL} \quad \dfrac{\dfrac{\Pi_{11}^{\dagger}}{\Gamma_2, \Delta_2 \vdash \Gamma_2'}}{\Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2', u{:}C}\,\mathrm{WlR}}{x{:}C \to D, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}\,{\to}\mathrm{L} \quad \dfrac{\Pi_6^{\dagger}}{y{:}A_1, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1}\,\Pi_6 \quad .$$

As another example, suppose that (25) has the form

$$\dfrac{\dfrac{\Pi_7}{\Delta_2, R\,x\,u\,v \vdash R\,0\,a\,x} \quad \dfrac{\Pi_8}{\Delta_2, R\,x\,u\,v \vdash R\,x\,u\,v}\,monr1}{\Delta_2, R\,x\,u\,v \vdash R\,a\,u\,v} \quad \dfrac{\dfrac{\dfrac{\dfrac{\Pi_{11}}{\Gamma_3, \Delta_3 \vdash \Gamma_3'}}{\Gamma_3, \Delta_3, R\,x\,u\,v \vdash \Gamma_3'}\,\mathrm{WrL} \quad \dfrac{\Pi_{12}}{\Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2', u{:}E}}{a{:}E \to (G \wedge H), \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}\,{\to}\mathrm{L} \quad \dfrac{\dfrac{\dfrac{\dfrac{\Pi_{13}}{\Gamma_2, \Delta_2 \vdash \Gamma_2'}}{\Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}\,\mathrm{WrL}}{v{:}H, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}\,\mathrm{WlL}}{v{:}G, v{:}H, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}\,\mathrm{WlL}}{v{:}G \wedge H, \Gamma_2, \Delta_2, R\,x\,u\,v \vdash \Gamma_2'}\,{\wedge}\mathrm{L}}{y{:}A_1, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1}\,\Pi_6\,{\to}\mathrm{L}$$

where $R\,x\,u\,v$ is not active in Π_6. In this case, we consider the rightmost branch and from it build

$$\dfrac{\dfrac{\dfrac{\Pi_{13}}{\Gamma_2, \Delta_2 \vdash \Gamma_2'}}{a{:}E \to (G \wedge H), \Gamma_2, \Delta_2 \vdash \Gamma_2'}\,\mathrm{WlL} \quad \dfrac{\Pi_6^{\dagger}}{y{:}A_1, \Gamma, \Delta, R\,x\,y\,z, R\,x\,u\,v \vdash \Gamma', z{:}B_1}\,\Pi_6 \quad .$$

Hence, with such suitable changes we obtain a proof Π_4'' of

$$y{:}A_1, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B_1\,.$$

This concludes the proof of the proposition. ■

Note that, although our sequents of the form $\Gamma, \Delta \vdash \Gamma'$ are multi-conclusioned (as opposed to the single-conclusioned sequents of propositional intuitionistic logic), in Definition 3.18 above we do not need to consider the branchings caused by $a{:}C \wedge D \in \Gamma'$, since by Fact 3.6 we have that: When the principal and active lwffs of a rule of S(B^+) appear on different sides of $\vdash$, then they are labelled differently. In particular, given Lemma 3.8, in an application of $\to$L or $\to$R the two active lwffs and the principal lwff all have distinct

labels. This implies that if, say, C is $(E \vee F) \to G$ and Γ' is $\{a{:}((E \vee F) \to G) \wedge D\}$, then if we apply the rules $\wedge$R and $\to$R we have

$$\dfrac{\dfrac{b{:}E \vee F, \Gamma, \Delta, R\,a\,b\,c \vdash c{:}G, x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n}{\Gamma, \Delta \vdash a{:}(E \vee F) \to G, x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n}\,{\to}\mathrm{R} \quad \Gamma, \Delta \vdash a{:}D, x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n}{\Gamma, \Delta \vdash a{:}((E \vee F) \to G) \wedge D, x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n}\,{\wedge}\mathrm{R} \quad .$$

By the condition on the application of $\to$R, b and c must be distinct labels not occurring elsewhere in the conclusion of the rule. In particular, b and c are not related to x. Hence, the branchings caused by applications of $\wedge$R are not dangerous for the satisfiability of the relevant analogue of the disjunction property, and we do not need to consider them.

Similarly, we do *not* need to define

- $a{:}A \vee B \in \mathfrak{B}(x, \Delta)$ if there exists a label b such that $\Delta \vdash R\,b\,a\,x$ is provable,
- $a{:}A \to B \in \mathfrak{B}(x, \Delta)$ if $a = x$ or there exist labels b and c such that $\Delta \vdash R\,b\,a\,c$ is provable in $\mathrm{S(B^+)}$ and $c{:}B \in \mathfrak{B}(x, \Delta)$.

nor

- Γ' is *x-branching with respect to Δ in* $\mathrm{S(B^+)}$ if $a{:}A \in \mathfrak{B}(x, \Delta)$ for some $a{:}A \in_{-} \Gamma'$.

From Proposition 3.19 we immediately obtain the following corollary.

Corollary 3.20 Let Π be an $\mathrm{S(B^+)}$-proof Π of the sequent

$$y_1{:}C_1, \ldots, y_m{:}C_m, \Gamma, \Delta, R\,x\,y_1\,z_1, \ldots, R\,x\,y_m\,z_m \vdash \Gamma', z_1{:}D_1, \ldots, z_m{:}D_m, x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n$$

where

(i) Γ is not x-branching with respect to Δ in $\mathrm{S(B^+)}$,
(ii) $x, y_1, \ldots, y_m, z_1, \ldots, z_m$ are all distinct.

Then there is an $\mathrm{S(B^+)}$-proof of either

$$\Gamma, \Delta \vdash \Gamma', x{:}A_i \to B_i \ \text{ for some } 1 \leq i \leq n\,,$$

or

$$y_j{:}C_j, \Gamma, \Delta, R\,x\,y_j\,z_j \vdash \Gamma', z_j{:}D_j \ \text{ for some } 1 \leq j \leq m\,.$$

■

This follows because, by (ii), for each j, the worlds x and y_j are compossible with respect to z_j, so that the sequent

$$y_1{:}C_1, \ldots, y_m{:}C_m, \Gamma, \Delta, R\,x\,y_1\,z_1, \ldots, R\,x\,y_m\,z_m \vdash \Gamma', z_1{:}D_1, \ldots, z_m{:}D_m, x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n$$

is equivalent (e.g. by m applications of $\to$R) to some sequent

$$\Gamma_1, \Delta \vdash \Gamma_1', x{:}C_1 \to D_1, \ldots, x{:}C_m \to D_m, x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n$$

and we conclude analogously to Proposition 3.19.

The intuition for Proposition 3.19 is that if in a backwards proof of

$$\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n$$

Γ is not x-branching with respect to Δ, as required by condition (16), then only one $x{:}A_i \to B_i$ (if any) will lead to axioms, i.e. all the other $x{:}A_j \to B_j$ are weak in the proof. The intuition for Corollary 3.20 is

analogous. In other words, the proposition and its corollary state that if there is no branching in x or in its 'predecessors', then we only need to follow one chain of worlds starting with x.

The condition that Γ is not x-branching with respect to Δ in $\mathrm{S(B^+)}$ in the proposition and the corollary (respectively condition (16) and condition (i)) requires that in Π there is neither an application of $\vee$L with the principal formula of the form $a{:}C \vee D$ where $a = x$ or $\Delta \vdash R\,a\,b\,x$ is provable for some b, nor an application of $\rightarrow$ L with principal formula of the form $a{:}C \rightarrow D$ where $\Delta \vdash R\,a\,b\,x$ is provable for some b. Thus the condition requires that in Π there is no implicit duplication of the lwffs $x{:}A_i \rightarrow B_i$.[6] If this condition is not satisfied, then we can immediately find counterexamples to the proposition. For example, like for propositional intuitionistic logic, although we have a proof of

$$x{:}(A_2 \rightarrow B_2) \vee (A_1 \rightarrow B_1) \vdash x{:}A_1 \rightarrow B_1, x{:}A_2 \rightarrow B_2\,,$$

neither of the two sequents

$$x{:}(A_2 \rightarrow B_2) \vee (A_1 \rightarrow B_1) \vdash x{:}A_i \rightarrow B_i\,,\ i \in \{1,2\}\,,$$

is provable. Similarly, as shown in the $\mathrm{S(B^+)}$-proof (28), we can prove the sequent

$$a{:}E \rightarrow (((A_2 \rightarrow B_2) \wedge C) \vee ((A_1 \rightarrow B_1) \wedge D)), c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \rightarrow B_1, x{:}A_2 \rightarrow B_2\,, \qquad (27)$$

but neither of the two sequents

$$a{:}E \rightarrow (((A_2 \rightarrow B_2) \wedge C) \vee ((A_1 \rightarrow B_1) \wedge D)), c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_i \rightarrow B_i\,,\ i \in \{1,2\}\,,$$

is provable. Proposition 3.19 fails for the sequent (27) since condition (16) is not satisfied:

$$a{:}E \rightarrow (((A_2 \rightarrow B_2) \wedge C) \vee ((A_1 \rightarrow B_1) \wedge D)) \in \mathfrak{B}(x, \Delta)$$

for $\Delta = \{R\,0\,a\,b, R\,b\,c\,x\}$, i.e. the lwff $a{:}E \rightarrow (((A_2 \rightarrow B_2) \wedge C) \vee ((A_1 \rightarrow B_1) \wedge D))$ is x-branching with respect to Δ, since $\Delta \vdash R\,a\,c\,x$ is provable by an application of *monr1*. Note, however, that the proof (28) contains subproofs of

$$x{:}(A_2 \rightarrow B_2) \wedge C, c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \rightarrow B_1, x{:}A_2 \rightarrow B_2$$

and

$$x{:}(A_1 \rightarrow B_1) \wedge D, c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \rightarrow B_1, x{:}A_2 \rightarrow B_2\,,$$

and since both satisfy condition (16) we can apply Proposition 3.19 to each of them separately. Indeed, we can find proofs of

$$x{:}(A_2 \rightarrow B_2) \wedge C, c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_2 \rightarrow B_2$$

and

$$x{:}(A_1 \rightarrow B_1) \wedge D, c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \rightarrow B_1\,.$$

As another example, the $\mathrm{S(B^+)}$-proof (29) shows that we can prove

$$a{:}(E \rightarrow (A_1 \rightarrow B_1)) \vee (E \rightarrow (A_2 \rightarrow B_2)), c{:}E, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \rightarrow B_1, x{:}A_2 \rightarrow B_2\,,$$

but neither

$$a{:}(E \rightarrow (A_1 \rightarrow B_1)) \vee (E \rightarrow (A_2 \rightarrow B_2)), c{:}E, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \rightarrow B_1$$

[6] For Proposition 3.19, and, similarly, for its corollary, we have that the implicit duplication resulting from an application of $\wedge$R does not affect the argument. Neither does the implicit duplication resulting from an application of $\vee$L or $\rightarrow$ L with principal formula $y{:}C \vee D$ or $y{:}C \rightarrow D$, where $\Delta \vdash R\,x\,y\,z$ or $\Delta \vdash R\,x\,z\,y$ is provable; this is because in a backwards proof this application of $\vee$L or $\rightarrow$ L can only occur after an application of $\rightarrow$ R with some $x{:}E \rightarrow F$ or some formula in Γ or Γ' as its principal formula.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta \vdash R\,a\,c\,x} \quad \dfrac{\dfrac{\begin{array}{c}\text{axioms}\\ \vdots \\ c{:}E \vdash c{:}E\end{array}}{c{:}E, c{:}F, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2, c{:}E}\ (\mathrm{W})}{c{:}E \wedge F, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2, c{:}E}\ \wedge\mathrm{L}}{a{:}E \to (((A_2 \to B_2) \wedge C) \vee ((A_1 \to B_1) \wedge D)), c{:}E \wedge F, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ \to\mathrm{L}}{a{:}E \to (((A_2 \to B_2) \wedge C) \vee ((A_1 \to B_1) \wedge D)), c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}(A_1 \to B_1) \vee (A_2 \to B_2)}\ \vee\mathrm{R}}{a{:}E \to (((A_2 \to B_2) \wedge C) \vee ((A_1 \to B_1) \wedge D)), R\,0\,a\,b \vdash b{:}(E \wedge F) \to ((A_1 \to B_1) \vee (A_2 \to B_2))}\ \to\mathrm{R}}{\vdash 0{:}(E \to (((A_2 \to B_2) \wedge C) \vee ((A_1 \to B_1) \wedge D))) \to ((E \wedge F) \to ((A_1 \to B_1) \vee (A_2 \to B_2)))}\ \to\mathrm{R} \qquad (28)$$

where $\Delta = \{R\,0\,a\,b, R\,b\,c\,x\}$, Π_1 is

$$\dfrac{\dfrac{\dfrac{}{R\,0\,a\,b \vdash R\,0\,a\,b}\ \mathrm{AXr}}{R\,0\,a\,b, R\,b\,c\,x \vdash R\,0\,a\,b}\ \mathrm{WrL} \quad \dfrac{\dfrac{}{R\,b\,c\,x \vdash R\,b\,c\,x}\ \mathrm{AXr}}{R\,0\,a\,b, R\,b\,c\,x \vdash R\,b\,c\,x}\ \mathrm{WrL}}{R\,0\,a\,b, R\,b\,c\,x \vdash R\,a\,c\,x}\ \mathit{monr1}$$

and Π_2 is

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axioms}\\ \vdots \\ x{:}A_2 \to B_2 \vdash x{:}A_2 \to B_2\end{array}}{x{:}A_2 \to B_2, x{:}C, c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ (\mathrm{W})}{x{:}(A_2 \to B_2) \wedge C, c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ \wedge\mathrm{L} \quad \dfrac{\dfrac{\begin{array}{c}\text{axioms}\\ \vdots \\ x{:}A_1 \to B_1 \vdash x{:}A_1 \to B_1\end{array}}{x{:}A_1 \to B_1, x{:}D, c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ (\mathrm{W})}{x{:}(A_1 \to B_1) \wedge D, c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ \wedge\mathrm{L}}{x{:}((A_2 \to B_2) \wedge C) \vee ((A_1 \to B_1) \wedge D), c{:}E \wedge F, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}\ \vee\mathrm{L}$$

.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1 \qquad\qquad \Pi_2}{a{:}E \to (A_1 \to B_1), c{:}E, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2 \quad a{:}E \to (A_2 \to B_2), c{:}E, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}\,\vee\mathrm{L}}{a{:}(E \to (A_1 \to B_1)) \vee (E \to (A_2 \to B_2)), c{:}E, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}}{a{:}(E \to (A_1 \to B_1)) \vee (E \to (A_2 \to B_2)), c{:}E, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}(A_1 \to B_1) \vee (A_2 \to B_2)}\,\vee\mathrm{R}}{a{:}(E \to (A_1 \to B_1)) \vee (E \to (A_2 \to B_2)), R\,0\,a\,b \vdash b{:}E \to ((A_1 \to B_1) \vee (A_2 \to B_2))}\,\to\mathrm{R}}{\vdash 0{:}((E \to (A_1 \to B_1)) \vee (E \to (A_2 \to B_2))) \to (E \to ((A_1 \to B_1) \vee (A_2 \to B_2)))}\,\to\mathrm{R} \tag{29}$$

where $\Delta = \{R\,0\,a\,b, R\,b\,c\,x\}$, Π_1 is

$$\dfrac{\Delta \vdash R\,a\,c\,x \quad \begin{array}{c}\text{axioms}\\ \vdots\\ c{:}E \vdash c{:}E\\ \vdots\\ (\mathrm{W})\end{array} \quad c{:}E, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2, c{:}E \qquad \begin{array}{c}\text{axioms}\\ \vdots\\ x{:}A_1 \to B_1 \vdash x{:}A_1 \to B_1\\ \vdots\\ (\mathrm{W})\end{array} \quad x{:}A_1 \to B_1, c{:}E, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}{a{:}E \to (A_1 \to B_1), c{:}E, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}\,\to\mathrm{L}$$

Π_2 is

$$\dfrac{\Delta \vdash R\,a\,c\,x \quad \begin{array}{c}\text{axioms}\\ \vdots\\ c{:}E \vdash c{:}E\\ \vdots\\ (\mathrm{W})\end{array} \quad c{:}E, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2, c{:}E \qquad \begin{array}{c}\text{axioms}\\ \vdots\\ x{:}A_2 \to B_2 \vdash x{:}A_2 \to B_2\\ \vdots\\ (\mathrm{W})\end{array} \quad x{:}A_2 \to B_2, c{:}E, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}{a{:}E \to (A_2 \to B_2), c{:}E, \Delta \vdash x{:}A_1 \to B_1, x{:}A_2 \to B_2}\,\to\mathrm{L}$$

and Π_3 is

$$\dfrac{\dfrac{\dfrac{}{R\,0\,a\,b \vdash R\,0\,a\,b}\,\mathrm{AXr}}{R\,0\,a\,b, R\,b\,c\,x \vdash R\,0\,a\,b}\,\mathrm{WrL} \qquad \dfrac{\dfrac{}{R\,b\,c\,x \vdash R\,b\,c\,x}\,\mathrm{AXr}}{R\,0\,a\,b, R\,b\,c\,x \vdash R\,b\,c\,x}\,\mathrm{WrL}}{R\,0\,a\,b, R\,b\,c\,x \vdash R\,a\,c\,x}\,\mathit{monr1} \quad .$$

nor

$$a{:}(E \to (A_1 \to B_1)) \vee (E \to (A_2 \to B_2)), c{:}E, R\,0\,a\,b, R\,b\,c\,x \vdash x{:}A_2 \to B_2$$

are provable.

In general, given an $\mathrm{S}(\mathrm{B}^+)$-proof Π of

$$\Gamma, \Delta \vdash \Gamma', x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n\,,$$

we can permute the applications of x-branching rules over the rules below them, provided that x-branching rules are applied at all in Π. We can thus transform Π so that it contains proofs of

$$\Gamma_1, \Delta \vdash \Gamma'_1, x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n \quad \text{and} \quad \cdots \quad \text{and} \quad \Gamma_m, \Delta \vdash \Gamma'_m, x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n$$

where Γ_j is not x-branching with respect to Δ in $\mathrm{S}(\mathrm{B}^+)$, for each $1 \leq j \leq m$. Then, for each

$$\Gamma_j, \Delta \vdash \Gamma'_j, x{:}A_1 \to B_1, \ldots, x{:}A_n \to B_n \qquad (1 \leq j \leq m)$$

there exists an i, with $1 \leq i \leq n$, such that there is an $\mathrm{S}(\mathrm{B}^+)$-proof of

$$\Gamma_j, \Delta \vdash \Gamma'_j, x{:}A_i \to B_i\,.$$

Performing this transformation, it is straightforward to show that applications of ClR with principal formula of the form $x{:}A \to B$ are eliminable in proofs of theorems of $\mathrm{S}(\mathrm{B}^+)$. Therefore, as another corollary of Proposition 3.19, we have that:

Corollary 3.21 Every sequent $\vdash 0{:}Z$ provable in $\mathrm{S}(\mathrm{B}^+)$ has a proof in which there are no applications of ClR with principal formula of the form $x{:}A \to B$. ■

To illustrate this, suppose that we have

$$\dfrac{\begin{array}{c}\Pi\\ \Gamma, \Delta \vdash \Gamma', x{:}A \to B, x{:}A \to B\end{array}}{\begin{array}{c}\Gamma, \Delta \vdash \Gamma', x{:}A \to B\\ \Pi_0\\ \vdash 0{:}Z\end{array}}\ \mathrm{ClR} \quad .$$

The corollary follows because, by applying the above transformation to Π, we permute possible applications of x-branching rules over the rules below them. Thus, we transform Π so that it contains proofs $\Pi_1, \ldots, \Pi_m$ of

$$\Gamma_1, \Delta \vdash \Gamma'_1, x{:}A \to B, x{:}A \to B \quad \text{and} \quad \cdots \quad \text{and} \quad \Gamma_m, \Delta \vdash \Gamma'_m, x{:}A \to B, x{:}A \to B\,,$$

i.e. we transform Π to

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1, \Delta \vdash \Gamma'_1, x{:}A \to B, x{:}A \to B\end{array} \quad \cdots \quad \begin{array}{c}\Pi_m\\ \Gamma_m, \Delta \vdash \Gamma'_m, x{:}A \to B, x{:}A \to B\end{array}}{\begin{array}{c}\Pi_{m+1}\\ \Gamma, \Delta \vdash \Gamma', x{:}A \to B, x{:}A \to B\\ \hline \Gamma, \Delta \vdash \Gamma', x{:}A \to B\\ \Pi_0\\ \vdash 0{:}Z\end{array}}\ \mathrm{ClR} \quad . \tag{30}$$

We can then permute the ClR upwards, i.e. we can further transform (30) to:

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Gamma_1, \Delta \vdash \Gamma'_1, x{:}A \to B, x{:}A \to B\end{array}}{\Gamma_1, \Delta \vdash \Gamma'_1, x{:}A \to B}\ \mathrm{ClR} \quad \cdots \quad \dfrac{\begin{array}{c}\Pi_m\\ \Gamma_m, \Delta \vdash \Gamma'_m, x{:}A \to B, x{:}A \to B\end{array}}{\Gamma_m, \Delta \vdash \Gamma'_m, x{:}A \to B}\ \mathrm{ClR}}{\begin{array}{c}\Pi^\dagger_{m+1}\\ \Gamma, \Delta \vdash \Gamma', x{:}A \to B\\ \Pi_0\\ \vdash 0{:}Z\end{array}}$$

where Γ_j is not x-branching with respect to Δ for each j such that $1 \leq j \leq m$. Then, for each such j we can find a proof of

$$\Gamma_j, \Delta \vdash \Gamma'_j, x{:}A \to B\,,$$

and thus eliminate the application of ClR with principal formula $x{:}A \to B$ displayed in (30). By iterating this for all other right contractions of formulas of the form $x{:}A \to B$ in Π_0, we obtain a proof of $\vdash 0{:}Z$ in which there are no applications of ClR with principal formula $x{:}A \to B$. In doing so, we replace explicit contractions of $x{:}A \to B$, via ClR, with implicit ones, i.e. the duplications of parametric formulas in the applications of $\vee$L or $\to$L with x-branching principal formulas.[7]

3.4 Contraction Elimination

Before proving the main result of this section, that all contractions of lwffs are eliminable in proofs of theorems of S(B$^+$), we introduce some more terminology that will be useful below. In an application of ClL or of ClR contracting $x{:}A$, we call $x{:}A$ a *contracted constituent*. We define the *rank* of such a contraction, in symbols $rank(x{:}A)$, to be the largest number of steps immediately above the conclusion of the contraction and containing at least one of the contracted formulas. Since we can always transform a proof so that the contraction of $x{:}A$ immediately follows the step introducing the second instance of $x{:}A$, the rank measures how many steps stand between the introduction of the first and second contracted constituent; thus the minimum possible rank is 2.

Theorem 3.22 Every sequent $\vdash 0{:}Z$ provable in S(B$^+$) has a proof in which there are no applications of the rules ClL and ClR.

Proof We adapt and extend the proof for the unlabelled sequent system for propositional classical logic given by Zeman in [37], and proceed by three nested inductions. The first induction is on the number of contractions in the S(B$^+$)-proof, the second on the grade of the contracted constituent, and the third on the rank of the contraction.

For the first induction, consider a proof Π of $\vdash 0{:}Z$ that contains $i+1$ contractions. Pick a 'highest' (i.e. uppermost in the leftmost branch) contraction in Π, i.e. consider a subproof of Π that ends with a contraction of $x{:}A$ and such that the proof above the contraction is contraction-free. By the permutability of the rules, we can assume, without loss of generality, that the contraction immediately follows the rule that introduces the second instance of $x{:}A$. We show, by induction on the grade of $x{:}A$, how to eliminate this contraction to obtain a proof Π' of $\vdash 0{:}Z$ that contains i contractions.

[$grade(x{:}A) = 0$] The base case, $grade(x{:}A) = 0$ is straightforward: since neither of the two contracted constituents $x{:}A$ can be introduced by a logical rule, and since only the first, at most, can be introduced by an axiom, the second $x{:}A$ must be introduced by a weakening or by an application of *monl*. In the first case, we simply delete the weakening and the contraction. In the second case, since $grade(x{:}A) = 0$, the proof must have the form:

$$\dfrac{\dfrac{\dfrac{\Pi_3}{\Delta \vdash R\,0\,a\,x} \qquad \dfrac{\dfrac{\Pi_1}{\Gamma_1, \Delta_1 \vdash \Gamma'_1, x{:}A}\,\mathrm{WlR}}{\Gamma_1, \Delta_1 \vdash \Gamma'_1, x{:}A, a{:}A}\;\dfrac{\Pi_2}{\Gamma, \Delta \vdash \Gamma', x{:}A, a{:}A}\;monl}{\dfrac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}A}{\Gamma, \Delta \vdash \Gamma', x{:}A}\,\mathrm{ClL}}}{\dfrac{\Pi_0}{\vdash 0{:}Z}}\,.$$

[7] Recall that, as we remarked in the introduction, the clear distinction between explicit and implicit contractions is one of the differences between our decision procedure for B$^+$ and that given in [16] based on the connection method.

We conclude by transforming this to:

$$\begin{array}{c}\Pi_1\\ \Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}A\\ \Pi_2^\dagger\\ \Gamma, \Delta \vdash \Gamma', x{:}A\\ \Pi_0\\ \vdash 0{:}Z\end{array} \quad .$$

Now let $grade(x{:}A) = k+1$. We proceed by induction on the rank of the contraction, $rank(x{:}A)$.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$] We consider the different rules introducing the second $x{:}A$, where, by Lemma 3.15 and Lemma 3.13, we do not need to consider the case when one of the two constituents $x{:}A$ is introduced by an axiom or by an application of *monl*.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \wedge C$, $\wedge$L] Suppose that $A = B \wedge C$ and that there is a contraction-free proof of $x{:}B \wedge C, x{:}B \wedge C, \Gamma, \Delta \vdash \Gamma'$. If one of the two instances of $x{:}B \wedge C$ is introduced by WlL, we conclude by deleting this WlL and ClL. Therefore suppose now that both instances of $x{:}B \wedge C$ are introduced by $\wedge$L, i.e.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x{:}B, x{:}C, x{:}B, x{:}C, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}B, x{:}C, x{:}B \wedge C, \Gamma, \Delta \vdash \Gamma'}\,\wedge\text{L}}{x{:}B \wedge C, x{:}B \wedge C, \Gamma, \Delta \vdash \Gamma'}\,\wedge\text{L}}{x{:}B \wedge C, \Gamma, \Delta \vdash \Gamma'}\,\text{ClL}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad .$$

We can apply the induction hypothesis to $x{:}B, x{:}C, x{:}B, x{:}C, \Gamma, \Delta \vdash \Gamma'$, to obtain a contraction-free proof Π_1' of $x{:}B, x{:}C, \Gamma, \Delta \vdash \Gamma'$, and then conclude by an application of $\wedge$L, i.e.

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1'\\ x{:}B, x{:}C, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}B \wedge C, \Gamma, \Delta \vdash \Gamma'}\,\wedge\text{L}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad .$$

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \wedge C$, $\wedge$R] Suppose that $A = B \wedge C$ and that there is a contraction-free proof of $\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}B \wedge C$. Since the contraction has rank 2, the first $x{:}B \wedge C$ is introduced twice, once into the left premise and once into the right premise of the application of $\wedge$R. We distinguish three cases.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \wedge C$, $\wedge$R, **case** 1] If the second $x{:}B \wedge C$ is introduced by WlR, or if both instances of the first $x{:}B \wedge C$ are introduced by WlR, then we conclude by simply deleting the weakening(s) and the ClR. For example, we transform

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Gamma, \Delta \vdash \Gamma', x{:}B \wedge C\end{array}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}B \wedge C}\,\text{WlR}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C}\,\text{ClR}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad \text{to} \quad \dfrac{\begin{array}{c}\Pi_1\\ \Gamma, \Delta \vdash \Gamma', x{:}B \wedge C\end{array}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad .$$

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \wedge C$, $\wedge$R, **case** 2] If one of the instances of the first $x{:}B \wedge C$ is introduced by WlR and the other by $\wedge$R, then we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, \Delta \vdash \Gamma', x{:}B}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}B}\,\mathrm{WlR} \quad \dfrac{\dfrac{\Pi_2}{\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}C} \quad \dfrac{\Pi_3}{\Gamma, \Delta \vdash \Gamma', x{:}C, x{:}C}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}C}\,\wedge\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}B \wedge C}\,\wedge\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C}\,\mathrm{ClR}}{\vdash 0{:}Z} \quad . \tag{31}$$

We then transform (31) by introducing the two immediate labelled subformulas of $x{:}B \wedge C$ with appropriate weakenings so that we can then apply $\wedge$R, i.e. we transform (31) to:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, \Delta \vdash \Gamma', x{:}B}}{\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}B}\,\mathrm{WlR} \quad \dfrac{\dfrac{\Pi_1}{\Gamma, \Delta \vdash \Gamma', x{:}B}}{\Gamma, \Delta \vdash \Gamma', x{:}C, x{:}B}\,\mathrm{WlR}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}B}\,\wedge\mathrm{R} \quad \dfrac{\dfrac{\Pi_2}{\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}C} \quad \dfrac{\Pi_3}{\Gamma, \Delta \vdash \Gamma', x{:}C, x{:}C}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}C}\,\wedge\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}B \wedge C}\,\wedge\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C}\,\mathrm{ClR}}{\vdash 0{:}Z} \quad .$$

We have thus transformed the proof so that both instances of $x{:}B \wedge C$ are introduced by $\wedge$R, and, since $grade(x{:}B)$ and $grade(x{:}C)$ are both less than $grade(x{:}B \wedge C)$, we can apply the induction hypothesis to

$$\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}B \quad \text{and} \quad \Gamma, \Delta \vdash \Gamma', x{:}C, x{:}C$$

to obtain contraction-free proofs Π_1' and Π_3' of

$$\Gamma, \Delta \vdash \Gamma', x{:}B \quad \text{and} \quad \Gamma, \Delta \vdash \Gamma', x{:}C\,,$$

respectively. Then, we conclude by an application of $\wedge$R, i.e.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1'}{\Gamma, \Delta \vdash \Gamma', x{:}B} \quad \dfrac{\Pi_3'}{\Gamma, \Delta \vdash \Gamma', x{:}C}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C}\,\wedge\mathrm{R}}{\vdash 0{:}Z} \quad .$$

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \wedge C$, $\wedge$R, **case** 3] If both instances of the first $x{:}B \wedge C$ are introduced by $\wedge$R, then the proof has the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}B} \quad \dfrac{\Pi_2}{\Gamma, \Delta \vdash \Gamma', x{:}C, x{:}B}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}B}\,\wedge\mathrm{R} \quad \dfrac{\dfrac{\Pi_3}{\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}C} \quad \dfrac{\Pi_4}{\Gamma, \Delta \vdash \Gamma', x{:}C, x{:}C}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}C}\,\wedge\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C, x{:}B \wedge C}\,\wedge\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}B \wedge C}\,\mathrm{ClR}}{\vdash 0{:}Z}$$

and we conclude, analogously to case 2, by applying the induction hypothesis.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \vee C$, $\vee$L] Suppose that $A = B \vee C$ and that there is a contraction-free proof of $x{:}B \vee C, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'$. Since the contraction has rank 2, the first $x{:}B \vee C$ is introduced twice, once into the left premise and once into the right premise of the application of $\vee$L. We distinguish three cases.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \vee C$, $\vee$L, **case** 1] If the second $x{:}B \vee C$ is introduced by WlL, or if both instances of the first $x{:}B \vee C$ are introduced by WlL, then we conclude by simply deleting the weakening(s) and the ClL. For example, we transform

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}B \vee C, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\text{WlL}}{x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\text{ClL}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \qquad \text{to} \qquad \dfrac{\begin{array}{c}\Pi_1\\ x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'\end{array}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad .$$

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \vee C$, $\vee$L, **case** 2] If one of the instances of the first $x{:}B \vee C$ is introduced by WlL and the other by $\vee$L, then we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x{:}B, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}B, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\text{WlL} \quad \dfrac{\begin{array}{c}\Pi_2\\ x{:}C, x{:}B, \Gamma, \Delta \vdash \Gamma'\end{array} \quad \begin{array}{c}\Pi_3\\ x{:}C, x{:}C, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}C, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\vee\text{L}}{x{:}B \vee C, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\vee\text{L}}{x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\text{ClL}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad . \tag{32}$$

We then transform (32) by introducing the two immediate labelled subformulas of $x{:}B \vee C$ with appropriate weakenings so that we can then apply $\vee$L, i.e. we transform (32) to

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x{:}B, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}B, x{:}B, \Gamma, \Delta \vdash \Gamma'}\,\text{WlL} \quad \dfrac{\begin{array}{c}\Pi_1\\ x{:}B, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}C, x{:}B, \Gamma, \Delta \vdash \Gamma'}\,\text{WlL}}{x{:}B, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\vee\text{L} \quad \dfrac{\begin{array}{c}\Pi_2\\ x{:}C, x{:}B, \Gamma, \Delta \vdash \Gamma'\end{array} \quad \begin{array}{c}\Pi_3\\ x{:}C, x{:}C, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}C, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\vee\text{L}}{x{:}B \vee C, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\vee\text{L}}{x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\text{ClL}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad .$$

We have thus transformed the proof so that both instances of $x{:}B \vee C$ are introduced by $\vee$L, and, since $grade(x{:}B)$ and $grade(x{:}C)$ are both less than $grade(x{:}B \vee C)$, we can apply the induction hypothesis to

$$x{:}B, x{:}B, \Gamma, \Delta \vdash \Gamma' \quad \text{and} \quad x{:}C, x{:}C, \Gamma, \Delta \vdash \Gamma'$$

to obtain contraction-free proofs Π_1' and Π_3' of

$$x{:}B, \Gamma, \Delta \vdash \Gamma' \quad \text{and} \quad x{:}C, \Gamma, \Delta \vdash \Gamma',$$

respectively. Then, we conclude by an application of $\vee$L, i.e.

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1'\\ x{:}B, \Gamma, \Delta \vdash \Gamma'\end{array} \quad \begin{array}{c}\Pi_3'\\ x{:}C, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\vee\text{L}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad .$$

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \vee C$, $\vee$L, **case** 3] If both instances of the first $x{:}B \vee C$ are introduced by $\vee$L, then the proof has the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x{:}B, x{:}B, \Gamma, \Delta \vdash \Gamma'\end{array} \quad \begin{array}{c}\Pi_2\\ x{:}C, x{:}B, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}B, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\vee\text{L} \quad \dfrac{\begin{array}{c}\Pi_3\\ x{:}C, x{:}B, \Gamma, \Delta \vdash \Gamma'\end{array} \quad \begin{array}{c}\Pi_4\\ x{:}C, x{:}C, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}C, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\vee\text{L}}{x{:}B \vee C, x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\vee\text{L}}{x{:}B \vee C, \Gamma, \Delta \vdash \Gamma'}\,\text{ClL}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}}$$

and we conclude, analogously to case 2, by applying the induction hypothesis.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \vee C$, $\vee$R] Suppose that $A = B \vee C$ and that there is a contraction-free proof of $\Gamma, \Delta \vdash \Gamma', x{:}B \vee C, x{:}B \vee C$. If one of the two instances of $x{:}B \vee C$ is introduced by WlR, we conclude by deleting this WlR and ClR. Therefore suppose now that both instances of $x{:}B \vee C$ are introduced by $\vee$R, i.e.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}C, x{:}B, x{:}C}\,\vee\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}C, x{:}B \vee C}\,\vee\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}B \vee C, x{:}B \vee C}\,\mathrm{ClR}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad .$$

We can apply the induction hypothesis to $\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}C, x{:}B, x{:}C$, to obtain a contraction-free proof Π_1' of $\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}C$, and then conclude by an application of $\vee$R, i.e.

$$\dfrac{\dfrac{\dfrac{\Pi_1'}{\Gamma, \Delta \vdash \Gamma', x{:}B, x{:}C}\,\vee\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}B \vee C}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad .$$

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \to C$, $\to$L] Suppose that $A = B \to C$ and that there is a contraction-free proof of $x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'$. Since the contraction has rank 2, the first $x{:}B \to C$ is introduced twice, once into the left premise and once into the right premise of the application of $\to$L. We distinguish three cases.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \to C$, $\to$L, **case** 1] If the second $x{:}B \to C$ is introduced by WlL, or if both instances of the first $x{:}B \to C$ are introduced by WlL, then we conclude by simply deleting the weakening(s) and the ClL. For example, we transform

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{WlL}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad \text{to} \quad \dfrac{\dfrac{\Pi_1}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad .$$

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \to C$, $\to$L, **case** 2] If one of the instances of the first $x{:}B \to C$ is introduced by WlL and the other by $\to$L, then we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_6}{\Delta \vdash R\,x\,y\,z} \quad \dfrac{\dfrac{\Pi_5}{\Gamma, \Delta \vdash \Gamma', y{:}B}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,\mathrm{WlL} \quad \dfrac{\dfrac{\Pi_2}{\Delta \vdash R\,x\,u\,v} \quad \dfrac{\Pi_3}{z{:}C, \Gamma, \Delta \vdash \Gamma', u{:}B} \quad \dfrac{\Pi_4}{v{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\to\mathrm{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\to\mathrm{L}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \quad . \qquad (33)$$

We then transform (33) by introducing the two immediate labelled subformulas of $x{:}B \to C$ with appropriate weakenings so that we can then apply $\to$L, i.e. we transform (33) to (34) or (35). In both cases, we have transformed the proof so that both instances of $x{:}B \to C$ are introduced by $\to$L, and we can proceed and conclude as in case 3.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \to C$, $\to$L, **case** 3] If both instances of the first $x{:}B \to C$ are introduced by $\to$L, then in the most general case the proof has the form shown in (36), where y, z, b, c, d, e

$$\dfrac{\dfrac{\Pi_6}{\Delta \vdash R\,x\,y\,z} \quad \dfrac{\dfrac{\Pi_6}{\Delta \vdash R\,x\,y\,z} \quad \dfrac{\dfrac{\Pi_5}{\Gamma, \Delta \vdash \Gamma', y{:}B}}{\Gamma, \Delta \vdash \Gamma', y{:}B, y{:}B}\,\mathrm{WlR}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,{\to}\mathrm{L} \quad \dfrac{\dfrac{\Pi_5}{\Gamma, \Delta \vdash \Gamma', y{:}B}}{z{:}C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,\mathrm{WrL} \quad \dfrac{\dfrac{\Pi_2}{\Delta \vdash R\,x\,u\,v} \quad \dfrac{\Pi_3}{z{:}C, \Gamma, \Delta \vdash \Gamma', u{:}B} \quad \dfrac{\Pi_4}{v{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL}}{\dfrac{\Pi_0}{\vdash 0{:}Z}} \quad . \tag{34}$$

$$\dfrac{\dfrac{\Pi_6}{\Delta \vdash R\,x\,y\,z} \quad \dfrac{\dfrac{\Pi_2}{\Delta \vdash R\,x\,u\,v} \quad \dfrac{\dfrac{\Pi_5}{\Gamma, \Delta \vdash \Gamma', y{:}B}}{\Gamma, \Delta \vdash \Gamma', y{:}B, u{:}B}\,\mathrm{WlR}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,{\to}\mathrm{L} \quad \dfrac{\dfrac{\Pi_5}{\Gamma, \Delta \vdash \Gamma', y{:}B}}{v{:}C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,\mathrm{WrL} \quad \dfrac{\dfrac{\Pi_2}{\Delta \vdash R\,x\,u\,v} \quad \dfrac{\Pi_3}{z{:}C, \Gamma, \Delta \vdash \Gamma', u{:}B} \quad \dfrac{\Pi_4}{v{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL}}{\dfrac{\Pi_0}{\vdash 0{:}Z}} \quad . \tag{35}$$

$$\dfrac{\dfrac{\Pi_8}{\Delta \vdash R\,x\,y\,z} \quad \dfrac{\dfrac{\Pi_2}{\Delta \vdash R\,x\,b\,c} \quad \dfrac{\Pi_3}{\Gamma, \Delta \vdash \Gamma', y{:}B, b{:}B} \quad \dfrac{\Pi_4}{c{:}C, \Gamma, \Delta \vdash \Gamma', y{:}B}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,{\to}\mathrm{L} \quad \dfrac{\dfrac{\Pi_5}{\Delta \vdash R\,x\,d\,e} \quad \dfrac{\Pi_6}{z{:}C, \Gamma, \Delta \vdash \Gamma', d{:}B} \quad \dfrac{\Pi_7}{e{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL}}{\dfrac{\Pi_0}{\vdash 0{:}Z}} \quad . \tag{36}$$

are all distinct.[8] In order to conclude, we need to show that we can transform Π_3 and Π_7 to Π_3' and Π_7' such that

$$\begin{array}{c}\Pi_3'\\ \Gamma,\Delta\vdash\Gamma',y{:}B\end{array}\quad\text{and}\quad\begin{array}{c}\Pi_7'\\ z{:}C,\Gamma,\Delta\vdash\Gamma'\end{array}$$

so that we can then apply $\to$L to obtain

$$\dfrac{\begin{array}{ccc}\Pi_8 & \Pi_3' & \Pi_7'\\ \Delta\vdash R\,x\,y\,z & \Gamma,\Delta\vdash\Gamma',y{:}B & z{:}C,\Gamma,\Delta\vdash\Gamma'\end{array}}{\begin{array}{c}x{:}B\to C,\Gamma,\Delta\vdash\Gamma'\\ \Pi_0\\ \vdash 0{:}Z\end{array}}\to\mathrm{L}\quad.$$

In other words, we must show that we can identify y and b (and d), and z and e (and c). This is not an easy task, so let us start by recalling what we already know. Fact 3.7 tells us that $x\neq 0$, since we are considering a proof of $\vdash 0{:}Z$. This means that

$$\begin{array}{c}\Pi_8\\ \Delta\vdash R\,x\,y\,z\end{array}\tag{37}$$

cannot follow by *iden* but must have one of the following forms:

(i) if $R\,x\,y\,z\in\Delta$, then (37) is

$$\dfrac{\dfrac{}{R\,x\,y\,z\vdash R\,x\,y\,z}\ \mathrm{AXr}}{\vdots\ \mathrm{WrL}\\ \Delta\vdash R\,x\,y\,z}\quad;$$

(ii) if $R\,x\,y\,z\notin\Delta$, then (37) is

$$\dfrac{\dfrac{}{R\,0\,x\,u\vdash R\,0\,x\,u}\ \mathrm{AXr}}{\vdots\ \mathrm{WrL}\\ \Delta\vdash R\,0\,x\,u}\qquad\begin{array}{c}\Pi_9\\ \Delta\vdash R\,u\,y\,z\end{array}\ monr1}{\Delta\vdash R\,x\,y\,z}$$

where $R\,0\,x\,u\in\Delta$.

Analogous observations hold for the forms of

$$\begin{array}{c}\Pi_2\\ \Delta\vdash R\,x\,b\,c\end{array}\quad\text{and}\quad\begin{array}{c}\Pi_5\\ \Delta\vdash R\,x\,d\,e\end{array}\quad.$$

As a consequence, there are six different cases that we must consider.

- Π_8, Π_2 and Π_5 all have the form (i).
- Π_8 and Π_2 have the form (i), and Π_5 has the form (ii); the symmetric case, where Π_8 and Π_5 have the form (i), and Π_2 has the form (ii), is dealt with analogously.
- Π_8 has the form (i), and Π_2 and Π_5 have the form (ii).
- Π_8 has the form (ii), and Π_2 and Π_5 have the form (i).
- Π_8 and Π_2 have the form (ii), and Π_5 has the form (i); the symmetric case, where Π_8 and Π_5 have the form (ii), and Π_2 has the form (i), is dealt with analogously.
- Π_8, Π_2 and Π_5 all have the form (ii).

[8] The least general case is when $y=b=d$ and $z=c=e$; then we conclude immediately by the induction hypotheses, since $grade(y{:}B)<grade(x{:}B\to C)$ and $grade(z{:}C)<grade(x{:}B\to C)$. All the cases 'in between', e.g. when $y=b\neq d$ and $z=c\neq e$, are dealt with like the most general case.

However, some of these cases are equivalent, since we can arbitrarily exchange Π_2, Π_5 and Π_8 in (36) to obtain equivalent proofs of $\vdash 0{:}Z$. Hence, there are basically just four distinct cases that we must consider:

- Case 3.1: Π_8, Π_2 and Π_5 all have the form (i).
- Case 3.2: Two of Π_8, Π_2 and Π_5 have the form (i), while the remaining one has the form (ii). As a concrete example, we consider the case when Π_8 and Π_2 have the form (i), and Π_5 has the form (ii).
- Case 3.3: One of Π_8, Π_2 and Π_5 has the form (i), while the other two have the form (ii). As a concrete example, we consider the case when Π_8 has the form (i), and Π_2 and Π_5 have the form (ii).
- Case 3.4: Π_8, Π_2 and Π_5 all have the form (ii).

We treat case 3.1 and case 3.4 first, and then the other two cases. In fact, we will show that only case 3.1 and case 3.4 may really occur in an $\mathrm{S}(\mathrm{B}^+)$-proof of $\vdash 0{:}Z$, while the other two may not.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \to C$, $\to$L, **case** 3.1] Suppose that $\Delta = \Delta_1 \cup \{R\,x\,y\,z, R\,x\,b\,c, R\,x\,d\,e\}$. Then (36) has the form shown in (38). Since (38) is a proof of $\vdash 0{:}Z$, the three rwffs $R\,x\,y\,z$, $R\,x\,b\,c$, and $R\,x\,d\,e$ are active in three applications of $\to$R in Π_0. Let $x{:}P_i \to Q_i$ for $1 \le i \le 3$ be the principal formulas of these applications of $\to$R, and, without loss of generality, assume that $\Gamma = \Gamma_1 \cup \{y{:}P_1, b{:}P_2, d{:}P_3\}$ and $\Gamma' = \Gamma'_1 \cup \{z{:}Q_1, c{:}Q_2, e{:}Q_3\}$. Then (38) has the form shown in (39). Proof (39) contains a contraction-free proof Π_8 of

$$x{:}B \to C, x{:}B \to C, \Gamma_1, y{:}P_1, b{:}P_2, d{:}P_3, \Delta_1, R\,x\,y\,z, R\,x\,b\,c, R\,x\,d\,e \vdash \Gamma'_1, z{:}Q_1, c{:}Q_2, e{:}Q_3\,,$$

and we can transform Π_8 so that it contains contraction-free proofs of

$$\begin{array}{c} x{:}B \to C, x{:}B \to C, \Gamma_1^1, y{:}P_1, b{:}P_2, d{:}P_3, \Delta_1, R\,x\,y\,z, R\,x\,b\,c, R\,x\,d\,e \vdash \Gamma_1^{1\prime}, z{:}Q_1, c{:}Q_2, e{:}Q_3 \\ \text{and} \quad \cdots \quad \text{and} \\ x{:}B \to C, x{:}B \to C, \Gamma_1^m, y{:}P_1, b{:}P_2, d{:}P_3, \Delta_1, R\,x\,y\,z, R\,x\,b\,c, R\,x\,d\,e \vdash \Gamma_1^{m\prime}, z{:}Q_1, c{:}Q_2, e{:}Q_3 \end{array}$$

where each Γ_1^j is not x-branching with respect to Δ_1 in $\mathrm{S}(\mathrm{B}^+)$. Then, by Corollary 3.20, for each

$$x{:}B \to C, x{:}B \to C, \Gamma_1^j, y{:}P_1, b{:}P_2, d{:}P_3, \Delta_1, R\,x\,y\,z, R\,x\,b\,c, R\,x\,d\,e \vdash \Gamma_1^{j\prime}, z{:}Q_1, c{:}Q_2, e{:}Q_3 \quad (1 \le j \le m)$$

there is a contraction-free $\mathrm{S}(\mathrm{B}^+)$-proof Π_9 of

$$x{:}B \to C, x{:}B \to C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1$$

or

$$x{:}B \to C, x{:}B \to C, \Gamma_1^j, b{:}P_2, \Delta_1, R\,x\,b\,c \vdash \Gamma_1^{j\prime}, c{:}Q_2$$

or

$$x{:}B \to C, x{:}B \to C, \Gamma_1^j, d{:}P_3, \Delta_1, R\,x\,d\,e \vdash \Gamma_1^{j\prime}, e{:}Q_3\,.$$

Let, for example, Π_9 be the proof of the first sequent. We have permuted possible applications of x-branching rules over the two consecutive applications of $\to$L (in both the middle and the rightmost branch). Therefore, Π_9 contains these two applications of $\to$L, i.e. Π_9 is:

$$\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}} \qquad \Pi_{10} \qquad \Pi_{13}}{\Delta_1, R\,x\,y\,z \vdash R\,x\,y\,z \quad x{:}B \to C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1, y{:}B \quad z{:}C, x{:}B \to C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1}\,{\to}\mathrm{L}$$
$$x{:}B \to C, x{:}B \to C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,x\,y\,z} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{R\,x\,b\,c \vdash R\,x\,b\,c}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,x\,b\,c} \qquad \dfrac{\Pi_3 \qquad \Pi_4}{\Gamma, \Delta \vdash \Gamma', y{:}B, b{:}B \qquad c{:}C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{R\,x\,d\,e \vdash R\,x\,d\,e}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,x\,d\,e} \qquad \dfrac{\Pi_6 \qquad \Pi_7}{z{:}C, \Gamma, \Delta \vdash \Gamma', d{:}B \qquad e{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta_1, R\,x\,y\,z, R\,x\,b\,c, R\,x\,d\,e \vdash \Gamma'}\,\mathrm{ClL}}{\begin{array}{c} x{:}B \to C, \Gamma, \Delta_1, R\,x\,y\,z, R\,x\,b\,c, R\,x\,d\,e \vdash \Gamma' \\ \Pi_0 \\ \vdash 0{:}Z \end{array}} \tag{38}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,x\,y\,z} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{R\,x\,b\,c \vdash R\,x\,b\,c}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,x\,b\,c} \qquad \dfrac{\Pi_3 \qquad \Pi_4}{\Gamma, \Delta \vdash \Gamma', y{:}B, b{:}B \qquad c{:}C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{R\,x\,d\,e \vdash R\,x\,d\,e}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,x\,d\,e} \qquad \dfrac{\Pi_6 \qquad \Pi_7}{z{:}C, \Gamma, \Delta \vdash \Gamma', d{:}B \qquad e{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, x{:}B \to C, \Gamma_1, y{:}P_1, b{:}P_2, d{:}P_3, \Delta_1, R\,x\,y\,z, R\,x\,b\,c, R\,x\,d\,e \vdash \Gamma_1', z{:}Q_1, c{:}Q_2, e{:}Q_3}\,\mathrm{ClL}}{\begin{array}{c} x{:}B \to C, \Gamma_1, y{:}P_1, b{:}P_2, d{:}P_3, \Delta_1, R\,x\,y\,z, R\,x\,b\,c, R\,x\,d\,e \vdash \Gamma_1', z{:}Q_1, c{:}Q_2, e{:}Q_3 \\ \Pi_0 \\ \vdash 0{:}Z \end{array}} \tag{39}$$

where Π_{10} is

$$\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta_1, R\,x\,y\,z \vdash R\,x\,y\,z} \qquad \dfrac{\Pi_{11} \qquad\qquad \Pi_{12}}{\Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1, y{:}B, y{:}B \qquad z{:}C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1, y{:}B}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1, y{:}B}$$

and Π_{13} is

$$\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta_1, R\,x\,y\,z \vdash R\,x\,y\,z} \qquad \dfrac{\Pi_{14} \qquad\qquad \Pi_{15}}{z{:}C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1, y{:}B \qquad z{:}C, z{:}C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1}\,{\to}\mathrm{L}}{z{:}C, x{:}B \to C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1}\,.$$

Since $grade(y{:}B) < grade(x{:}B \to C)$ and $grade(z{:}C) < grade(x{:}B \to C)$, by the induction hypotheses, there is a contraction-free proof Π'_{11} of

$$\Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1, y{:}B\,,$$

and a contraction-free proof Π'_{15} of

$$z{:}C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1\,.$$

We can thus replace Π_9 with:

$$\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta_1, R\,x\,y\,z \vdash R\,x\,y\,z} \qquad \dfrac{\Pi'_{11} \qquad\qquad \Pi'_{15}}{\Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1, y{:}B \qquad z{:}C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma_1^j, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma_1^{j\prime}, z{:}Q_1}\,.$$

Proceeding analogously for each $1 \le j \le m$, we obtain a contraction-free proof Π'_8 of

$$x{:}B \to C, \Gamma_1, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma'_1, z{:}Q_1$$

from which we conclude by weakening:

$$\dfrac{\dfrac{\dfrac{\Pi'_8}{x{:}B \to C, \Gamma_1, y{:}P_1, \Delta_1, R\,x\,y\,z \vdash \Gamma'_1, z{:}Q_1}}{\vdots\,(\mathrm{W})}}{x{:}B \to C, x{:}B \to C, \Gamma_1, y{:}P_1, b{:}P_2, d{:}P_3, \Delta_1, R\,x\,y\,z, R\,x\,b\,c, R\,x\,d\,e \vdash \Gamma'_1, z{:}Q_1, c{:}Q_2, e{:}Q_3} \qquad \dfrac{\Pi_0}{\vdash 0{:}Z}$$

To summarize, analogously to Corollary 3.21, we have first copied $x{:}B \to C$ in each branch via the implicit contractions in applications of x-branching rules, and then eliminated the explicit contraction.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \to C$, $\to$L, **case** 3.4] Suppose that $R\,x\,y\,z \notin \Delta$, $R\,x\,b\,c \notin \Delta$, and $R\,x\,d\,e \notin \Delta$. Then (36) has the form shown in (40), where Π_8, Π_2 and Π_5 all end with an applications of *monr1*, that is Π_8 is

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,u \vdash R\,0\,x\,u}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,0\,x\,u} \qquad \dfrac{\Pi_9}{\Delta \vdash R\,u\,y\,z}\,\mathit{monr1}}{\Delta \vdash R\,x\,y\,z}$$

$$\dfrac{\dfrac{\dfrac{\Pi_8}{\Delta \vdash R\,x\,y\,z} \quad \dfrac{\dfrac{\Pi_2}{\Delta \vdash R\,x\,b\,c} \quad \dfrac{\Pi_3}{\Gamma, \Delta \vdash \Gamma', y{:}B, b{:}B} \quad \dfrac{\Pi_4}{c{:}C, \Gamma, \Delta \vdash \Gamma', y{:}B}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,{\to}\mathrm{L} \quad \dfrac{\dfrac{\Pi_5}{\Delta \vdash R\,x\,d\,e} \quad \dfrac{\Pi_6}{z{:}C, \Gamma, \Delta \vdash \Gamma', d{:}B} \quad \dfrac{\Pi_7}{e{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL}}{\dfrac{\Pi_0}{\vdash 0{:}Z}} \quad . \tag{40}$$

Π_2 is

$$\dfrac{\dfrac{}{R\,0\,x\,v \vdash R\,0\,x\,v}\ \text{AXr}\quad\vdots\ \text{WrL}\quad \Delta \vdash R\,0\,x\,v \qquad \overset{\Pi_{10}}{\Delta \vdash R\,v\,b\,c}}{\Delta \vdash R\,x\,b\,c}\ \mathit{monr1}$$

Π_5 is

$$\dfrac{\dfrac{}{R\,0\,x\,w \vdash R\,0\,x\,w}\ \text{AXr}\quad\vdots\ \text{WrL}\quad \Delta \vdash R\,0\,x\,w \qquad \overset{\Pi_{11}}{\Delta \vdash R\,w\,d\,e}}{\Delta \vdash R\,x\,d\,e}\ \mathit{monr1}$$

where $\Delta = \Delta_1 \cup \{R\,0\,x\,u, R\,0\,x\,v, R\,0\,x\,w\}$. Since we are considering a proof of $\vdash 0{:}Z$, it must be the case that $u = v = w$; otherwise we would need three applications of $\to$R with active rwffs $R\,0\,x\,u$, $R\,0\,x\,v$, and $R\,0\,x\,w$, where, however, the side conditions would not be satisfied (because x should be an arbitrary label in all three rwffs). Thus, let $u = v = w$, so that Π_2 and Π_5 are

$$\dfrac{\dfrac{}{R\,0\,x\,u \vdash R\,0\,x\,u}\ \text{AXr}\quad\vdots\ \text{WrL}\quad \Delta \vdash R\,0\,x\,u \qquad \overset{\Pi_{10}}{\Delta \vdash R\,u\,b\,c}}{\Delta \vdash R\,x\,b\,c}\ \mathit{monr1} \qquad \text{and} \qquad \dfrac{\dfrac{}{R\,0\,x\,u \vdash R\,0\,x\,u}\ \text{AXr}\quad\vdots\ \text{WrL}\quad \Delta \vdash R\,0\,x\,u \qquad \overset{\Pi_{11}}{\Delta \vdash R\,u\,d\,e}}{\Delta \vdash R\,x\,d\,e}\ \mathit{monr1}$$

where $\Delta = \Delta_1 \cup \{R\,0\,x\,u\}$.

Consider now Π_9; by Lemma 3.11, we know that it must be the case that $R\,u\,y\,z \in \Delta_1$ and thus $R\,u\,y\,z \in \Delta$. Analogous observations hold for Π_{10} and Π_{11}, and thus it must be the case that $\{R\,u\,y\,z, R\,u\,b\,c, R\,u\,d\,e\} \subseteq \Delta_1 \subset \Delta$. We can now conclude as in the previous case, case 3.1, by reasoning on the applications of $\to$R that have these three rwffs as active rwffs.

[$\mathit{grade}(x{:}A) = k+1$, $\mathit{rank}(x{:}A) = 2$, $A = B \to C$, $\to$**L, case** 3.2] Suppose that $\Delta = \Delta_1 \cup \{R\,0\,x\,u, R\,x\,y\,z, R\,x\,b\,c\}$. Then (36) has the form (41). Since (41) is a proof of $\vdash 0{:}Z$, the two rwffs $R\,x\,y\,z$ and $R\,x\,b\,c$ are active in two applications of $\to$R in Π_0. Let $x{:}P_1 \to Q_1$ and $x{:}P_2 \to Q_2$ be the principal formulas of these applications of $\to$R, and, without loss of generality, assume that $\Gamma = \Gamma_1 \cup \{y{:}P_1, b{:}P_2\}$ and $\Gamma' = \Gamma'_1 \cup \{z{:}Q_1, c{:}Q_2\}$. (We conclude analogously when $y = b$ and $z = c$.) Similarly, the rwff $R\,0\,x\,u$ is active in an application of $\to$R in Π_0, say with principal formula $0{:}P_3 \to Q_3$. Since this application of $\to$R must follow the other two, for otherwise the condition on $\to$R would not be satisfied, and since by Fact 3.16 we can assume that all applications of WrL occur immediately below the axioms, (41) has the form (42).

Consider now Π_9. By Lemma 3.11, it must be the case that $R\,u\,d\,e \in \Delta_1$ and thus $R\,u\,d\,e \in \Delta$. That is, $\{R\,0\,x\,u, R\,x\,y\,z, R\,x\,b\,c, R\,u\,d\,e\} \subseteq \Delta_1 \subset \Delta$. Consider now again the two lowest applications of $\to$R in (42). By Fact 3.6, $x{:}P_2 \to Q_2$ cannot be a subformula of $x{:}P_3$, and thus, since x does not occur in $\Gamma_4, \Delta_4 \vdash \Gamma'_4, 0{:}P_3 \to Q_3$, the lwff $x{:}P_2 \to Q_2$ must be a subformula of $u{:}Q_3$; e.g. Q_3 is $D \vee (P_2 \to Q_2)$, $\Gamma_3 = \{x{:}P_3\} \cup \Gamma_4$, $\Delta_3 = \Delta_4$, $\Gamma'_3 = \{u{:}D\} \cup \Gamma'_4$, and Π^3_0 is

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,u \vdash R\,0\,x\,u}\ \text{AXr}\quad\vdots\ \text{WrL}\quad\vdots\quad \Delta_4, R\,0\,x\,u \vdash R\,0\,x\,u \qquad x{:}P_3, \Gamma_4, \Delta_4, R\,0\,x\,u \vdash \Gamma'_4, u{:}D, x{:}P_2 \to Q_2}{x{:}P_3, \Gamma_4, \Delta_4, R\,0\,x\,u \vdash \Gamma'_4, u{:}D, u{:}P_2 \to Q_2}\ \mathit{monl}}{x{:}P_3, \Gamma_4, \Delta_4, R\,0\,x\,u \vdash \Gamma'_4, u{:}D \vee (P_2 \to Q_2)}\ \vee\text{R}\ .$$

An analogous observation holds also for $x{:}P_1 \to Q_1$. However, Lemma 3.13 tells us that we can transform each proof of $\vdash 0{:}Z$ so that all applications of *monl* have an active lwff of grade 0. (In fact, (41) and (42) should already have this form.) In particular, when the active formula of *monl* is introduced by an application of $\to$R, we can push monotony from lwffs to rwffs (cf. the proof of Lemma 3.13). But this contradicts our initial assumption that $\{R\,0\,x\,u, R\,x\,b\,c\} \subset \Delta$, so that this case (case 3.2) cannot occur.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,x\,y\,z} \qquad \dfrac{\Pi^1}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B} \qquad \dfrac{\Pi^2}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{\dfrac{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL}}$$
$$\Pi_0$$
$$\vdash 0{:}Z \tag{41}$$

where Π^1 is

$$\dfrac{\dfrac{\dfrac{}{R\,x\,b\,c \vdash R\,x\,b\,c}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,x\,b\,c} \qquad \dfrac{\Pi_3}{\Gamma, \Delta \vdash \Gamma', y{:}B, b{:}B} \qquad \dfrac{\Pi_4}{c{:}C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,{\to}\mathrm{L}$$
$$x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B$$

and Π^2 is

$$\dfrac{\dfrac{\dfrac{\dfrac{}{R\,0\,x\,u \vdash R\,0\,x\,u}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,0\,x\,u} \qquad \dfrac{\Pi_9}{\Delta \vdash R\,u\,d\,e}\,monr1 \qquad \dfrac{\Pi_6}{z{:}C, \Gamma, \Delta \vdash \Gamma', d{:}B} \qquad \dfrac{\Pi_7}{e{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{\Delta \vdash R\,x\,d\,e}$$
$$z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'$$

.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}}{\vdots\ \mathrm{WrL}}}{\Delta \vdash R\,x\,y\,z} \quad \overset{\Pi^1}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B} \quad \overset{\Pi^2}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}}{x{:}B \to C, x{:}B \to C, \Gamma_1, y{:}P_1, b{:}P_2, \Delta_1, R\,0\,x\,u, R\,x\,y\,z, R\,x\,b\,c \vdash \Gamma'_1, z{:}Q_1, c{:}Q_2}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma_1, y{:}P_1, b{:}P_2, \Delta_1, R\,0\,x\,u, R\,x\,y\,z, R\,x\,b\,c \vdash \Gamma'_1, z{:}Q_1, c{:}Q_2}\,\mathrm{ClL}}{\overset{\Pi^1_0}{y{:}P_1, b{:}P_2, \Gamma_2, \Delta_2, R\,0\,x\,u, R\,x\,y\,z, R\,x\,b\,c \vdash \Gamma'_2, z{:}Q_1, c{:}Q_2}}\,{\to}\mathrm{R}}{\overset{\Pi^2_0}{b{:}P_2, \Gamma_2, \Delta_2, R\,0\,x\,u, R\,x\,b\,c \vdash \Gamma'_2, x{:}P_1 \to Q_1, c{:}Q_2}}\,{\to}\mathrm{R}}{\overset{\Pi^3_0}{\Gamma_3, \Delta_3, R\,0\,x\,u \vdash \Gamma'_3, x{:}P_2 \to Q_2}}\,{\to}\mathrm{R}}{\overset{\Pi^4_0}{\Gamma_4, \Delta_4 \vdash \Gamma'_4, 0{:}P_3 \to Q_3}}\,{\to}\mathrm{R}}{\vdash 0{:}Z} \tag{42}$$

where Π^1 is

$$\dfrac{\dfrac{\dfrac{}{R\,x\,b\,c \vdash R\,x\,b\,c}\,\mathrm{AXr}}{\vdots\ \mathrm{WrL}}}{\Delta \vdash R\,x\,b\,c} \quad \overset{\Pi_3}{\Gamma, \Delta \vdash \Gamma', y{:}B, b{:}B} \quad \overset{\Pi_4}{c{:}C, \Gamma, \Delta \vdash \Gamma', y{:}B}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,{\to}\mathrm{L}$$

and Π^2 is

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,u \vdash R\,0\,x\,u}\,\mathrm{AXr}}{\vdots\ \mathrm{WrL}}}{\Delta \vdash R\,0\,x\,u} \quad \overset{\Pi_9}{\Delta \vdash R\,u\,d\,e}}{\Delta \vdash R\,x\,d\,e}\,monr1 \quad \overset{\Pi_6}{z{:}C, \Gamma, \Delta \vdash \Gamma', d{:}B} \quad \overset{\Pi_7}{e{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}\ .$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,x\,y\,z} \qquad \dfrac{\Pi^1 \qquad\qquad \Pi^2}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B \quad z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'} \to \mathrm{L}}{\dfrac{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL}}}{\begin{array}{c}\Pi_0 \\ \vdash 0{:}Z\end{array}} \tag{43}$$

where Π^1 is

$$\dfrac{\dfrac{\dfrac{}{R\,0\,x\,u \vdash R\,0\,x\,u}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,0\,x\,u} \qquad \dfrac{\Pi_9 \qquad\qquad \Pi_3 \qquad\qquad \Pi_4}{\Delta \vdash R\,u\,b\,c \quad \Gamma, \Delta \vdash \Gamma', y{:}B, b{:}B \quad c{:}C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,monr1}{\Delta \vdash R\,x\,b\,c} \to \mathrm{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B}$$

and Π^2 is

$$\dfrac{\dfrac{\dfrac{\dfrac{}{R\,0\,x\,v \vdash R\,0\,x\,v}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,0\,x\,v} \qquad \dfrac{\Pi_{10} \qquad\qquad \Pi_6 \qquad\qquad \Pi_7}{\Delta \vdash R\,v\,d\,e \quad z{:}C, \Gamma, \Delta \vdash \Gamma', d{:}B \quad e{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}\,monr1}{\Delta \vdash R\,x\,d\,e} \to \mathrm{L}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,.$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,x\,y\,z} \qquad \dfrac{\Pi^1}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B} \qquad \dfrac{\Pi^2}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}}{x{:}B \to C, x{:}B \to C, \Gamma_1, y{:}P_1, \Delta_1, R\,0\,x\,u, R\,x\,y\,z \vdash \Gamma_1', z{:}Q_1}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma_1, y{:}P_1, \Delta_1, R\,0\,x\,u, R\,x\,y\,z \vdash \Gamma_1', z{:}Q_1}\,\mathrm{ClL}}{\dfrac{\dfrac{\dfrac{\Pi_0^1}{y{:}P_1, \Gamma_2, \Delta_2, R\,0\,x\,u, R\,x\,y\,z \vdash \Gamma_2', z{:}Q_1}}{\Gamma_2, \Delta_2, R\,0\,x\,u \vdash \Gamma_2', x{:}P_1 \to Q_1}\,{\to}\mathrm{R}}{\dfrac{\dfrac{\Pi_0^2}{x{:}P_2, \Gamma_3, \Delta_3, R\,0\,x\,u \vdash \Gamma_3', u{:}Q_2}}{\Gamma_3, \Delta_3 \vdash \Gamma_3', 0{:}P_2 \to Q_2}\,{\to}\mathrm{R}}}{\dfrac{\Pi_0^3}{\vdash 0{:}Z}} \tag{44}
$$

where Π^1 is

$$
\dfrac{\dfrac{\dfrac{}{R\,0\,x\,u \vdash R\,0\,x\,u}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,0\,x\,u} \qquad \dfrac{\Pi_9}{\Delta \vdash R\,u\,b\,c}\ monr1 \qquad \dfrac{\Pi_3}{\Gamma, \Delta \vdash \Gamma', y{:}B, b{:}B} \qquad \dfrac{\Pi_4}{c{:}C, \Gamma, \Delta \vdash \Gamma', y{:}B}
$$
$$
\dfrac{\Delta \vdash R\,x\,b\,c}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,{\to}\mathrm{L}
$$

and Π^2 is

$$
\dfrac{\dfrac{\dfrac{}{R\,0\,x\,u \vdash R\,0\,x\,u}\,\mathrm{AXr}}{\vdots\,\mathrm{WrL}}}{\Delta \vdash R\,0\,x\,u} \qquad \dfrac{\Pi_{10}}{\Delta \vdash R\,u\,d\,e}\ monr1 \qquad \dfrac{\Pi_6}{z{:}C, \Gamma, \Delta \vdash \Gamma', d{:}B} \qquad \dfrac{\Pi_7}{e{:}C, z{:}C, \Gamma, \Delta \vdash \Gamma'}
$$
$$
\dfrac{\Delta \vdash R\,x\,d\,e}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}\ .
$$

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \to C$, $\to$L, **case** 3.3] Suppose that $\Delta = \Delta_1 \cup \{R\,0\,x\,u, R\,0\,x\,v, R\,x\,y\,z\}$. Then (36) has the form (43). Since (43) is a proof of $\vdash 0{:}Z$, the three rwffs $R\,0\,x\,u$, $R\,0\,x\,v$ and $R\,x\,y\,z$ are active in applications of $\to$R in Π_0, and for this to be possible, it must be the case that $v = u$. Hence, let $x{:}P_1 \to Q_1$ and $0{:}P_2 \to Q_2$ be the principal formulas of these two applications of $\to$R, and, without loss of generality, assume that $\Gamma = \Gamma_1 \cup \{y{:}P_1\}$ and $\Gamma' = \Gamma'_1 \cup \{z{:}Q_1\}$. Then (43) has the form (44).

Consider now Π_9 and Π_{10}. By Lemma 3.11, it must be the case that $\{R\,0\,x\,u, R\,x\,y\,z, R\,u\,b\,c, R\,u\,d\,e\} \subseteq \Delta_1 \subset \Delta$. As in case 3.2 we can then argue that $x{:}P_1 \to Q_1$ must be a subformula of $u{:}Q_2$, and then follow again the same reasoning to argue that this case is not possible.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = 2$, $A = B \to C$, $\to$R] Suppose that $A = B \to C$ and that there is a contraction-free proof Π_1 of $\Gamma, \Delta \vdash \Gamma', x{:}B \to C, x{:}B \to C$ such that

$$\dfrac{\begin{array}{c}\Pi_1\\ \Gamma, \Delta \vdash \Gamma', x{:}B \to C, x{:}B \to C\end{array}}{\begin{array}{c}\Gamma, \Delta \vdash \Gamma', x{:}B \to C\\ \Pi_0\\ \vdash 0{:}Z\end{array}}\ \text{ClR} \quad .$$

Then, by Corollary 3.21, there is a contraction-free proof Π'_1 of $\Gamma, \Delta \vdash \Gamma', x{:}B \to C$, and we conclude easily:

$$\begin{array}{c}\Pi'_1\\ \Gamma, \Delta \vdash \Gamma', x{:}B \to C\\ \Pi_0\\ \vdash 0{:}Z\end{array} \quad .$$

This concludes the proof for the case $grade(x{:}A) = k+1$ and $rank(x{:}A) = 2$. Consider now the final case when $grade(x{:}A) = k+1$ and $rank(x{:}A) = j+1 > 2$. In this case, it is possible that the first contracted constituent was introduced by a weakening into one or more of the places that give the contraction a rank of $j+1$. We make the mode of introduction of that constituent uniform by replacing all such introductions by weakening with introductions by the proper logical rule; the rest of the proof is as before, modulo possible applications of weakening and AXr. For example, given the proof

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Delta \vdash R\,x\,y\,z\end{array} \quad \dfrac{\begin{array}{c}\Pi_2\\ \Gamma_1, \Delta_1 \vdash \Gamma'_1\end{array}}{x{:}B \to C, \Gamma_1, \Delta_1 \vdash \Gamma'_1}\ \text{WlL} \quad \begin{array}{c}\Pi_3\\ x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B\end{array}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\ \to\text{L} \quad \dfrac{\begin{array}{c}\Pi_4\\ \Gamma_2, \Delta_2 \vdash \Gamma'_2\end{array}}{x{:}B \to C, \Gamma_2, \Delta_2 \vdash \Gamma'_2}\ \text{WlL} \quad \begin{array}{c}\Pi_5\\ z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'\end{array}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\ \to\text{L}}{\begin{array}{c}x{:}B \to C, \Gamma, \Delta \vdash \Gamma'\\ \Pi_0\\ \vdash 0{:}Z\end{array}}\ \text{ClL} \tag{45}$$

we transform the middle branch to

$$\dfrac{\begin{array}{c}\Pi_6\\ \Delta_1 \vdash R\,x\,y\,z\end{array} \quad \dfrac{\begin{array}{c}\Pi_2\\ \Gamma_1, \Delta_1 \vdash \Gamma'_1\end{array}}{\Gamma_1, \Delta_1 \vdash \Gamma'_1, y{:}B}\ \text{WlR} \quad \dfrac{\begin{array}{c}\Pi_2\\ \Gamma_1, \Delta_1 \vdash \Gamma'_1\end{array}}{z{:}C, \Gamma_1, \Delta_1 \vdash \Gamma'_1}\ \text{WlL}}{\begin{array}{c}x{:}B \to C, \Gamma_1, \Delta_1 \vdash \Gamma'_1\\ \Pi_3\\ x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B\end{array}}\ \to\text{L}$$

and the rightmost branch to

$$\dfrac{\begin{array}{c}\Pi_7\\ \Delta_2 \vdash R\,x\,y\,z\end{array} \quad \dfrac{\begin{array}{c}\Pi_4\\ \Gamma_2, \Delta_2 \vdash \Gamma'_2\end{array}}{\Gamma_2, \Delta_2 \vdash \Gamma'_2, y{:}B}\ \text{WlR} \quad \dfrac{\begin{array}{c}\Pi_4\\ \Gamma_2, \Delta_2 \vdash \Gamma'_2\end{array}}{z{:}C, \Gamma_2, \Delta_2 \vdash \Gamma'_2}\ \text{WlL}}{\begin{array}{c}x{:}B \to C, \Gamma_2, \Delta_2 \vdash \Gamma'_2\\ \Pi_5\\ z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'\end{array}}\ \to\text{L} \quad .$$

(We have here employed Fact 3.16, which tells us that we can always permute applications of WrL upwards so that they immediately follow the axioms; hence Π_6 and Π_7 exist, and are the proofs of $\Delta_1 \vdash R\,x\,y\,z$ and $\Delta_2 \vdash R\,x\,y\,z$, respectively.) The rank of ClL is still $j+1$, but now we have a proof in which all highest introductions of the first contracted constituent are by the rule proper to that constituent. We consider the different cases for this proper rule.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = j+1$, $A = B \wedge C$, $\wedge$L] We conclude straightforwardly by first permuting the uppermost $\wedge$L over the rule below it to obtain a proof with $rank(x{:}B \wedge C) = j$, and then applying the induction hypothesis.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = j+1$, $A = B \wedge C$, $\wedge$R] We conclude straightforwardly by first permuting the uppermost $\wedge$R over the rule below it to obtain a proof with $rank(x{:}B \wedge C) = j$, and then applying the induction hypothesis.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = j+1$, $A = B \vee C$, $\vee$L] We conclude straightforwardly by first permuting the uppermost $\vee$L over the rule below it to obtain a proof with $rank(x{:}B \vee C) = j$, and then applying the induction hypothesis.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = j+1$, $A = B \vee C$, $\vee$R] We conclude straightforwardly by first permuting the uppermost $\vee$R over the rule below it to obtain a proof with $rank(x{:}B \vee C) = j$, and then applying the induction hypothesis.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = j+1$, $A = B \to C$, $\to$L] We consider a contraction-free proof of

$$x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'$$

such that

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta_1 \vdash R\,x\,b\,c} \quad \dfrac{\Pi_2}{\Gamma_1, \Delta_1 \vdash \Gamma_1', b{:}B} \quad \dfrac{\Pi_3}{c{:}C, \Gamma_1, \Delta_1 \vdash \Gamma_1'}}{x{:}B \to C, \Gamma_1, \Delta_1 \vdash \Gamma_1'} \to\text{L} \quad \dfrac{\dfrac{\Pi_5}{\Delta_2 \vdash R\,x\,d\,e} \quad \dfrac{\Pi_6}{\Gamma_2, \Delta_2 \vdash \Gamma_2', d{:}B} \quad \dfrac{\Pi_7}{e{:}C, \Gamma_2, \Delta_2 \vdash \Gamma_2'}}{x{:}B \to C, \Gamma_2, \Delta_2 \vdash \Gamma_2'} \to\text{L}}{\dfrac{\dfrac{\dfrac{\Pi_9}{\Delta \vdash R\,x\,y\,z} \quad \dfrac{\Pi_4}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B} \quad \dfrac{\Pi_8}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'} \to\text{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'} \text{ClL}}{\vdash 0{:}Z} \tag{46}$$

where, by Fact 3.16, we can safely assume that Π_4 and Π_8 do not contain applications of WrL.

As explained for Lemma 3.5, we cannot, in general, reduce the rank of the contraction by permuting the uppermost applications of $\to$L over the rules below them. For example, suppose that the highest rule application in Π_4 is an application of $\to$R with active rwff $R\,x\,b\,c$; i.e. with the same active rwff as the $\to$L immediately above Π_4. Then the middle branch of (46) is

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta_3, R\,x\,b\,c \vdash R\,x\,b\,c} \quad \dfrac{\Pi_2}{b{:}P_1, \Gamma_3, \Delta_3, R\,x\,b\,c \vdash \Gamma_3', c{:}Q_1, b{:}B} \quad \dfrac{\Pi_3}{c{:}C, b{:}P_1, \Gamma_3, \Delta_3, R\,x\,b\,c \vdash \Gamma_3', c{:}Q_1}}{x{:}B \to C, b{:}P_1, \Gamma_3, \Delta_3, R\,x\,b\,c \vdash \Gamma_3', c{:}Q_1} \to\text{L}}{x{:}B \to C, \Gamma_3, \Delta_3 \vdash \Gamma_3', x{:}P_1 \to Q_1} \to\text{R}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B}$$

with $\Gamma_1 = \{b{:}P_1\} \cup \Gamma_3$, $\Delta_1 = \Delta_3 \cup \{R\,x\,b\,c\}$, and $\Gamma_1' = \Gamma_3' \cup \{c{:}Q_1\}$. It immediately follows that we cannot permute the application of $\to$L over the application of $\to$R.

An analogous observation holds for the rightmost branch. If the highest rule application in Π_8 is an application of $\to$R with active rwff $R\,x\,d\,e$ (i.e. with the same active rwff as the $\to$L immediately above Π_8), then the rightmost branch of (46) is

$$\dfrac{\dfrac{\Pi_5 \qquad \Pi_6 \qquad\qquad \Pi_7}{\Delta_4, R\,x\,d\,e \vdash R\,x\,d\,e \quad d{:}P_2, \Gamma_4, \Delta_4, R\,x\,d\,e \vdash \Gamma_4', e{:}Q_2, d{:}B \quad e{:}C, d{:}P_2, \Gamma_4, \Delta_4, R\,x\,d\,e \vdash \Gamma_4', e{:}Q_2}\,{\to}\mathrm{L}}{\dfrac{x{:}B \to C, d{:}P_2, \Gamma_4, \Delta_4, R\,x\,d\,e \vdash \Gamma_4', e{:}Q_2}{x{:}B \to C, \Gamma_4, \Delta_4 \vdash \Gamma_4', x{:}P_2 \to Q_2}\,{\to}\mathrm{R}}$$
$$\Pi_{11}$$
$$z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'$$

with $\Gamma_2 = \{d{:}P_2\} \cup \Gamma_4$, $\Delta_2 = \Delta_4 \cup \{R\,x\,d\,e\}$, and $\Gamma_2' = \Gamma_4' \cup \{e{:}Q_2\}$, and we cannot permute the application of $\to$L over the application of $\to$R.

When the two cases occur simultaneously, (46) has the form shown in (47). (We conclude analogously when the two cases do not occur simultaneously.) Although we cannot permute the uppermost applications of $\to$L over the applications of $\to$R below them, we can permute these applications of $\to$R over the second, lowest, application of $\to$L in (47). This is possible because $x{:}P_1 \to Q_1$ and $y{:}B$, and $x{:}P_2 \to Q_2$ and $z{:}C$ are independent, in the sense that it cannot be that (reasoning backwards) $x{:}P_1 \to Q_1$ follows from $y{:}B$, or that $x{:}P_2 \to Q_2$ follows from $z{:}C$. In the middle branch, $x{:}P_1 \to Q_1$ must then follow from some formula in Γ or Γ', and Π_{10} can be divided into two separate subproofs, Π_{10}' and Π_{10}''; the first yields $y{:}B$, and $x{:}P_1 \to Q_1$ is active in the second. (If $y{:}B$ is already contained in Γ_3, then Π_{10}' is empty and $\Pi_{10}'' = \Pi_{10}$.) Similarly, in the rightmost branch, $x{:}P_2 \to Q_2$ must follow from some formula in Γ or Γ', and Π_{11} can be divided into two separate subproofs, Π_{11}' and Π_{11}''; the first yields $z{:}C$, and $x{:}P_2 \to Q_2$ is active in the second. (If $z{:}C$ is already contained in Γ_4, then Π_{11}' is empty and $\Pi_{11}'' = \Pi_{11}$.)

Given these separations, we can transform (47) by possibly by renaming some labels and permuting the applications of $\to$R downwards to obtain a proof in which $rank(x{:}B \to C) \leq j$, and then conclude by the induction hypothesis.

We give an example of this transformation where, for simplicity, we assume that $P_1 \to Q_1 = P_2 \to Q_2$, $\Gamma' = \{x{:}P_1 \to Q_1\}$, and that Π_{10} and Π_{11} are both empty, so that $R\,x\,b\,c = R\,x\,d\,e$, $\Gamma_3 = \Gamma$, $\Delta_3 = \Delta_4 = \Delta$, $\Gamma_3' = \Gamma' \cup \{y{:}B\} = \{x{:}P_1 \to Q_1, y{:}B\}$, $\Gamma_4 = \Gamma \cup \{z{:}C\}$, and $\Gamma_4' = \Gamma' = \{x{:}P_1 \to Q_1\}$. Then (47) has the form shown in (48) which we can transform to (49) so that the rank is reduced and we can conclude by the induction hypothesis.

[$grade(x{:}A) = k+1$, $rank(x{:}A) = j+1$, $A = B \to C$, $\to$R] We consider a contraction-free proof of $\Gamma, \Delta \vdash \Gamma', x{:}B \to C, x{:}B \to C$ such that:

$$\dfrac{\dfrac{\dfrac{\Pi_1}{u{:}B, \Gamma_1, \Delta_1, R\,x\,u\,v \vdash \Gamma_1', v{:}C}\,{\to}\mathrm{R}}{\Gamma_1, \Delta_1 \vdash \Gamma_1', x{:}B \to C}$$
$$\dfrac{\dfrac{\Pi_2}{y{:}B, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}C, x{:}B \to C}\,{\to}\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', x{:}B \to C, x{:}B \to C}\,\mathrm{ClR}}{\Gamma, \Delta \vdash \Gamma', x{:}B \to C}$$
$$\Pi_0$$
$$\vdash 0{:}Z$$

Since $x{:}B \to C$ is a parametric formula in Π_2, we can postpone its introduction, i.e., by Lemma 3.5, we permute the uppermost $\to$R over the rule below it to obtain a contraction-free proof of rank j (we might then need to rename some labels; e.g. if the first rule in Π_2 is a weakening introducing a formula with u or v). We conclude by the induction hypothesis. This also concludes the proof of the theorem. ■

4 Complexity of Proof Search

We show the decidability of S(B$^+$) by combining bounds on contraction and monotony rules, which bound the number of labelled formulas that can appear in an S(B$^+$)-proof Π of $\vdash 0{:}Z$, with bounds on the number

$$\dfrac{\dfrac{\dfrac{\Pi_9}{\Delta \vdash R\,x\,y\,z} \quad \dfrac{\Pi^1}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B} \quad \dfrac{\Pi^2}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL}}{\vdash 0{:}Z}\,\Pi_0 \tag{47}$$

where Π^1 is

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta_3, R\,x\,b\,c \vdash R\,x\,b\,c} \quad \dfrac{\Pi_2}{b{:}P_1, \Gamma_3, \Delta_3, R\,x\,b\,c \vdash \Gamma_3', c{:}Q_1, b{:}B} \quad \dfrac{\Pi_3}{c{:}C, b{:}P_1, \Gamma_3, \Delta_3, R\,x\,b\,c \vdash \Gamma_3', c{:}Q_1}}{x{:}B \to C, b{:}P_1, \Gamma_3, \Delta_3, R\,x\,b\,c \vdash \Gamma_3', c{:}Q_1}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma_3, \Delta_3 \vdash \Gamma_3', x{:}P_1 \to Q_1}\,{\to}\mathrm{R}}{x{:}B \to C, \Gamma, \Delta \vdash \Gamma', y{:}B}\,\Pi_{10}$$

and Π^2 is

$$\dfrac{\dfrac{\dfrac{\Pi_5}{\Delta_4, R\,x\,d\,e \vdash R\,x\,d\,e} \quad \dfrac{\Pi_6}{d{:}P_2, \Gamma_4, \Delta_4, R\,x\,d\,e \vdash \Gamma_4', e{:}Q_2, d{:}B} \quad \dfrac{\Pi_7}{e{:}C, d{:}P_2, \Gamma_4, \Delta_4, R\,x\,d\,e \vdash \Gamma_4', e{:}Q_2}}{x{:}B \to C, d{:}P_2, \Gamma_4, \Delta_4, R\,x\,d\,e \vdash \Gamma_4', e{:}Q_2}\,{\to}\mathrm{L}}{x{:}B \to C, \Gamma_4, \Delta_4 \vdash \Gamma_4', x{:}P_2 \to Q_2}\,{\to}\mathrm{R}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash \Gamma'}\,\Pi_{11}\ .$$

$$\dfrac{\dfrac{\dfrac{\Pi_9 \quad \Pi^1 \quad \Pi^2}{\Delta \vdash R\,x\,y\,z \quad x{:}B \to C, \Gamma, \Delta \vdash x{:}P_1 \to Q_1, y{:}B \quad z{:}C, x{:}B \to C, \Gamma, \Delta \vdash x{:}P_1 \to Q_1}\,{\to}\mathrm{L}}{x{:}B \to C, x{:}B \to C, \Gamma, \Delta \vdash x{:}P_1 \to Q_1}\,\mathrm{ClL}}{\begin{array}{c}\Pi_0\\ \vdash 0{:}Z\end{array}} \tag{48}$$

where Π^1 is

$$\dfrac{\dfrac{\Pi_1 \quad \Pi_2 \quad \Pi_3}{\Delta, R\,x\,b\,c \vdash R\,x\,b\,c \quad b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1, y{:}B, b{:}B \quad c{:}C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1, y{:}B}\,{\to}\mathrm{L}}{x{:}B \to C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1, y{:}B}\,{\to}\mathrm{R}}{x{:}B \to C, \Gamma, \Delta \vdash x{:}P_1 \to Q_1, y{:}B}$$

and Π^2 is

$$\dfrac{\dfrac{\Pi_1 \quad \Pi_6 \quad \Pi_7}{\Delta, R\,x\,b\,c \vdash R\,x\,b\,c \quad z{:}C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1, b{:}B \quad c{:}C, z{:}C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1}\,{\to}\mathrm{L}}{z{:}C, x{:}B \to C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1}\,{\to}\mathrm{R}}{z{:}C, x{:}B \to C, \Gamma, \Delta \vdash x{:}P_1 \to Q_1}\,{\to}\mathrm{L} \quad .$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\Pi_9}{\Delta \vdash R\,x\,y\,z} \quad \overset{\Pi^1}{x{:}B \to C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1, y{:}B} \quad \overset{\Pi^2}{z{:}C, x{:}B \to C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1}}{x{:}B \to C, x{:}B \to C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1}\,{\to}\mathrm{L}}{x{:}B \to C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1}\,\mathrm{ClL}}{x{:}B \to C, \Gamma, \Delta \vdash x{:}P_1 \to Q_1}\,{\to}\mathrm{R}}{\overset{\Pi_0}{\vdash 0{:}Z}} \tag{49}$$

where Π^1 is

$$\dfrac{\overset{\Pi_1}{\Delta, R\,x\,b\,c \vdash R\,x\,b\,c} \quad \overset{\Pi_2}{b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1, y{:}B, b{:}B} \quad \overset{\Pi_3}{c{:}C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1, y{:}B}}{x{:}B \to C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1, y{:}B}\,{\to}\mathrm{L}$$

and Π^2 is

$$\dfrac{\overset{\Pi_1}{\Delta, R\,x\,b\,c \vdash R\,x\,b\,c} \quad \overset{\Pi_6}{z{:}C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1, b{:}B} \quad \overset{\Pi_7}{c{:}C, z{:}C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1}}{z{:}C, x{:}B \to C, b{:}P_1, \Gamma, \Delta, R\,x\,b\,c \vdash c{:}Q_1}\,{\to}\mathrm{L} \quad .$$

of labels and relational formulas that can be generated in Π. These bounds, combined with the soundness and completeness of $\mathrm{S}(\mathrm{B}^+)$, tell us that the provability (validity) problem for the logic B^+ is decidable in $O(n \log n)$-space.

Let us begin by summarizing the above results, from which it follows that to prove $\vdash 0{:}Z$ in $\mathrm{S}(\mathrm{B}^+)$ it suffices to search for a proof in the restriction of $\mathrm{S}(\mathrm{B}^+)$ that consists of: the axiom AXl restricted to atomic lwffs and the axiom AXr; the logical rules for $\wedge$, $\vee$ and $\rightarrow$; the weakening rules; and the monotony rules

$$\frac{\Delta \vdash R\,0\,x\,y \quad \Gamma, \Delta \vdash \Gamma', x{:}A}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ \mathit{monl} \qquad \text{and} \qquad \frac{\Delta \vdash R\,0\,x\,a \quad \Delta \vdash R\,a\,y\,z}{\Delta \vdash R\,x\,y\,z}\ \mathit{monr1},$$

where by Lemma 3.12 and Lemma 3.13 *monl* is such that A is atomic, $x{:}A$ is not introduced by *monl* and $R\,0\,x\,y \in \Delta$, and by Lemma 3.10 and Lemma 3.11 *monr1* is such that $R\,0\,a\,x \in \Delta$ and $R\,x\,y\,z \in \Delta$.

When we embarked on our proof-theoretical analysis of $\mathrm{S}(\mathrm{B}^+)$, we did so aiming at bounding the rules that do not simplify sequents, so that no infinite branches may arise and proof search terminates. The bounds that we established in § 3 allow us to define a measure that decreases at *almost* every rule application. Let us namely define the *degree of an lwff* $x{:}A$ to be twice the number of symbols in the string representation of A, i.e. $2 \times |A|$, and the *degree of a sequent* S to be the sum of the degrees of the lwffs in S plus the number of rwffs in S. Then we have:

Fact 4.1 In each rule of (the restriction of) $\mathrm{S}(\mathrm{B}^+)$, other than *monl* and *monr1*, each premise has smaller degree than the conclusion. ■

The rules *monl* and *monr1* are exceptions because they do not reduce the number of connectives or rwffs. For these rules we must then reason differently. Although applications of *monr1* generate sequents of the same size as the conclusion, they do not generate new rwffs since only labels and rwffs currently in existence can be employed. Moreover, Lemma 3.11 tells us that there need not be two consecutive applications of *monr1*. Thus, the provability of $\Delta \vdash R\,x\,y\,z$ in $\mathrm{S}(\mathrm{B}^+)$ reduces to checking either if $R\,x\,y\,z \in \Delta$, or, in the case of an application of *monr1*, if there is a label a such that $R\,0\,x\,a \in \Delta$ and $R\,a\,y\,z \in \Delta$. Both problems are solvable in space linear in the *size* of the end-sequent $S = \vdash 0{:}Z$, which we denote by $|S|$.

Applications of *monl* are similarly bounded, in that *monl* need not be applied twice to the same (atomic) formula, even though labelled differently, in the same branch.

As a consequence, in each branch of a proof of $\vdash 0{:}Z$, leading from the conclusion to an axiom, there can be at most one application of *monl* or of *monr1*, depending on whether the axiom is AXl or AXr.[9] This, together with Fact 4.1, implies that every proof of a sequent $S = \vdash 0{:}Z$ is bounded in length by the degree of S, which is in turn linearly bounded in terms of its size $n = |S|$, plus the linear space required for applications of *monl* and *monr1*. That is, any proof of $S = \vdash 0{:}Z$ will have depth $O(n)$, where $n = |S|$.

In Fact 3.2 we observed that $\mathrm{S}(\mathrm{B}^+)$ has a subformula property, so that when searching for a proof of $\vdash 0{:}Z$ we only need to look for sequents that contain subformulas of Z. However, subformulas may be labelled differently. While new rwffs can be generated by applications of $\rightarrow$L or $\rightarrow$R, new labels can only be generated by applications of $\rightarrow$R, since applications of $\rightarrow$L and *monl* label subformulas with different (currently in existence) labels. (Similarly, the active rwffs of *monr1* must already exist.) In particular, each application of $\rightarrow$R generates two new labels, so that the number of labels in a proof is linearly bounded above in any branch in terms of the number of $\rightarrow$ operators appearing in the end-sequent S, i.e. $O(n)$. Similarly, the number of lwffs and rwffs in any generated sequent is also $O(n)$.

These results give rise to a proof procedure that uses $O(n \log n)$-space. To illustrate this, we begin by distinguishing two different kinds of branching in the search space for proofs.

[9] To specify this, we could also introduce a *monotony index* on the principal formula of *monl*, i.e.

$$\frac{\Delta \vdash R\,0\,x\,y \quad \Gamma, \Delta \vdash \Gamma', (x{:}A)^0}{\Gamma, \Delta \vdash \Gamma', (y{:}A)^1}\ \mathit{monl},$$

with the convention that *monl* can only be applied when $y{:}A$ has index 1, and that, reasoning forwards, when we introduce $x{:}A$ we set its index to 0. To accommodate possibly longer proofs, we then generalize the definitions of the degree function: we define the *degree of an lwff* $(x{:}A)^m$, i.e. $x{:}A$ with monotony index m, to be $2 \times |A| + m$. Note that the degree as defined above is a special case of this, since $m = 0$ for each $x{:}A$; in both cases, the size of $(x{:}A)^m$ is bounded by $O(|S|)$ since m is constant (1 or 0).

- *Conjunctive branching:* applying rules with multiple premises builds a branching tree, where all branches must be proved.
- *Disjunctive branching:* more than one rule may be applicable and a given rule may be applicable in different ways.

Conjunctive branching, caused by rules like $\wedge$R or $\to$L, leads to proofs that, if stored in their entirety, are exponential in size. Disjunctive branching arises when more than one rule may be applied to a sequent or when a single rule can be applied in different ways (e.g. a weakening). This is reflected by a branching point in the search space for proofs, rather than in proofs themselves.[10]

We base our decision procedure on the adaptation and extension of standard ideas, e.g. [18, 19]. Rather than storing entire proofs (branches), we store a sequent and a stack that maintains information sufficient to reconstruct branching points. Each stack entry is a triple:

(i) the name of the rule applied,
(ii) the principal formula of the sequent, and
(iii) an index.

The index records sufficient information such that on return to the branching point another branch, if one remains, can be generated. In other words, the index allows us to reconstruct the sequent associated with a branching point by replaying the stack entries to that point. For example, for $\to$L the index records which premise we choose, the first, second or third premise.

A proof begins with the end-sequent $S = \vdash 0{:}Z$ and the empty stack. Each rule application (we assume rules are ordered somehow and we apply them in order) generates a new sequent and appropriately extends the stack. There are two possibilities.

If the generated sequent is an axiom, and the stack contains no conjunctive branching points that still need to be explored, then S is provable. Otherwise we pop entries off the stack until we find a conjunctive branching point that must be further explored and then generate the next branch (first incrementing the index on the stack to record this).

Alternatively, if we arrive at a sequent that is not an axiom and no rule applies, then we pop stack entries and continue at the first available disjunctive branching point with the choices that remain, e.g. apply a different weakening, determined by the index, or apply the next available rule. If no such branching point remains, then S is not provable.

Note that $\to$L introduces, as well as conjunctive branching, also disjunctive branching: if the principal formula is $x{:}B \to C$, any labels x_1 and x_2 currently in existence are possible candidates for producing $x_1{:}B$ and $x_2{:}C$ in the generated sequents. We use the index to record which labels we choose so that on backtracking we can generate the next possible candidates; we can determine what the possible candidates are by looking into the stack and examining previous applications of $\to$R, WrL and *monr1*. Moreover, since $\to$L is not invertible, it might be that $\to$L simply does not lead to axioms. In this case, rules other than $\to$L must be tried on backtracking.

This procedure terminates and is complete. Termination follows because the stack depth (i.e. the proof depth) is $O(n)$ and the branching is bounded (by n, the number of rules, and the maximum number of premises). Completeness follows straightforwardly from the soundness and completeness of S(B$^+$), Theorem 2.2, and the fact that the entire search space is navigated, if need be, on backtracking.

To compute the space requirement of the procedure, we simply need to sum the requirements of the stack and of generated sequents. (Strictly speaking, we should also add the space required to solve relational queries of the form $\Delta \vdash R\,x\,y\,z$, but we argued above that these queries reduce to a checking problem that is solvable in linear space.)

Storing the stack requires $O(n \log n)$-space. This is obtained by multiplying the stack depth, $O(n)$, with the space required by each stack entry, which is $O(log\, n)$-space. The latter bound results as follows. (i) The names of the rules require constant space. (ii) We can store each of the $O(n)$ labels using $O(log\, n)$ bits.

[10] Note that disjunctive branching is related to the invertibility of the rules, where we call a rule *invertible* when the provability of the conclusion implies the provability of the premise(s). By induction on the lengths of proofs it follows that all the rules of (the restricted system) S(B$^+$) other than $\to$L, WrL, WlL, WlR, *monl* and *monr1* are invertible. However, if we are interested only in proofs of theorems $\vdash 0{:}Z$, then all rules other than $\to$L, WrL, WlL and WlR are invertible.

Similarly, we can represent each of the $O(n)$ subformulas by an index into the end-sequent S, which requires only $O(log\, n)$ bits. Thus, storing the principal formula of each rule application requires $O(log\, n)$ bits. (iii) The index recording branching points requires $O(log\, n)$-space as well.

Above we argued that the bounds on the number of labels, lwffs and rwffs in any generated sequent are $O(n)$, like the bound on the depth of proofs. Since we can represent each of the $O(n)$ labels using $O(log\, n)$ bits, each of the $O(n)$ rwffs can be represented using $O(log\, n)$ bits as well. Since also each of the $O(n)$ lwffs can be represented using $O(log\, n)$ bits, it follows that any generated sequent requires $O(n \log n)$-space.

Thus, our decision procedure requires $O(n \log n)$-space, and since a formula Z of B^+ is provable iff the sequent $\vdash 0{:}Z$ is provable in $\mathrm{S}(\mathrm{B}^+)$, we have:

Theorem 4.2 Provability for the basic positive relevance logic B^+ is decidable in $O(n \log n)$-space. ■

5 Conclusions and Future Work

We have given an $O(n \log n)$-space decision procedure for the basic positive relevance logic B^+ by extending the proof-theoretical method of [35] and showing how the results fit and refine the abstract recipe of [6]. More specifically, we first introduced a sound and complete, cut-free, labelled sequent system $\mathrm{S}(\mathrm{B}^+)$, and then investigated and optimized $\mathrm{S}(\mathrm{B}^+)$ to eliminate some of its rules or establish bounds on their applications.

This paper is part of ongoing research that aims at developing a framework for applying proof theory to the analysis of complexity of families of non-classical logics, and thereby designing decision procedures for these logics [4, 5, 6, 35]. As future work, we plan to use our proof-theoretical method to investigate the complexity of other relevance logics, and thereby try to solve some of the decidability problems that are still open in this area. Current work includes extending our approach to the basic relevance logic B, which we obtain by extending B^+ with negation, whose addition raises a number of interesting questions.

To begin with, since there are different ways of dealing with negation in relevance logics, e.g. [11, 12, 26, 27], we must first choose the one that best suits our needs. For example, in [5, 35] we gave natural deduction systems for relevance logics with negation by defining a unary function $*$ on worlds such that $\vDash a{:}\neg A$ iff $\nvDash a^*{:}A$. (Informally, a^* is the world that does not deny what a asserts, i.e. a and a^* are 'compatible' worlds.) Assuming that we follow this choice, we are then faced with other decisions. For example, we could formalize the logic B in terms of the cut-free labelled sequent system that we obtain by extending $\mathrm{S}(\mathrm{B}^+)$ with the logical rules for negation

$$\frac{\Gamma, \Delta \vdash \Gamma', x^*{:}A}{x{:}\neg A, \Gamma, \Delta \vdash \Gamma'}\ \neg\mathrm{L} \qquad \text{and} \qquad \frac{x^*{:}A, \Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma', x{:}\neg A}\ \neg\mathrm{R}\,,$$

and with the relational rules

$$\frac{\Delta \vdash R\,0\,x\,y}{\Delta \vdash R\,0\,y^*\,x^*}\ \textit{antitonic}\,, \quad \frac{}{\vdash R\,0\,x\,x^{**}}\ {**}\textit{intuitionistic}\,, \quad \frac{}{\vdash R\,0\,x^{**}\,x}\ {**}\textit{classical}\,.$$

Or, alternatively, we could follow Priest and Sylvan [23] and Restall [24], and develop a cut-free labelled sequent system for B in terms of the *reduced semantics* they propose. This would, for example, require us to distinguish two pairs of logical rules for relevant implication, depending on whether the lwff $x{:}A \to B$ they introduce has label $x \neq 0$ or $x = 0$. That is,

$$\frac{\Delta \vdash R\,x\,y\,z \quad \Gamma, \Delta \vdash \Gamma', y{:}A \quad z{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \to B, \Gamma, \Delta \vdash \Gamma'}\ \to\mathrm{L}\ (x \neq 0)\,, \qquad \frac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \to B}\ \to\mathrm{R}\ (x \neq 0)\,,$$

$$\frac{\Gamma, \Delta \vdash \Gamma', a{:}A \quad a{:}B, \Gamma, \Delta \vdash \Gamma'}{0{:}A \to B, \Gamma, \Delta \vdash \Gamma'}\ \to\mathrm{L}_0\,, \qquad \frac{a{:}A, \Gamma, \Delta \vdash \Gamma', a{:}B}{\Gamma, \Delta \vdash \Gamma', 0{:}A \to B}\ \to\mathrm{R}_0\,,$$

where, in $\to\mathrm{R}$, y and z are distinct and do not occur in $\Gamma, \Delta \vdash \Gamma', x{:}A \to B$, while $\to\mathrm{R}_0$ has the side condition that a does not occur in $\Gamma, \Delta \vdash \Gamma', 0{:}A \to B$.

This distinction on $x{:}A \to B$ has the effect of eliminating rwffs of the form $R\,0\,x\,y$, and thus also applications of *iden*. This has the advantage that it simplifies the solution of relational queries $\Delta \vdash R\,a\,b\,c$.

However, it also has the disadvantage that it does not allow us anymore to define the partial order $\sqsubseteq$ in terms or R and 0, but requires us to introduce $\sqsubseteq$ explicitly, and reason explicitly about its properties. For example, *monl* becomes

$$\frac{\Delta \vdash x \sqsubseteq y \quad \Gamma, \Delta \vdash \Gamma', x{:}A}{\Gamma, \Delta \vdash \Gamma', y{:}A}\ monl,$$

and $\sqsubseteq$ is formalized by the rules

$$\frac{}{\vdash x \sqsubseteq x} \qquad \text{and} \qquad \frac{\Delta \vdash x \sqsubseteq y \quad \Delta \vdash y \sqsubseteq z}{\Delta \vdash x \sqsubseteq z}\ .$$

We are currently investigating all these possibilities for dealing with negation.

Future work will also involve the implementation of our decision procedure and the empirical evaluation of its time performance. To this end, observe that in [35] we show that our framework is such that the resulting labelled natural deduction systems can be straightforwardly implemented in the generic theorem prover Isabelle [21]; a similar observation applies for labelled sequent systems. Our Isabelle encodings provide a simple and natural environment for interactive proof development that supports hierarchical structuring, and we plan to implement our decision procedures to automate theorem proving for B^+ and other non-classical logics.

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