Lower Bounds For Randomized Searching on m Rays*

Institut für Informatik — Report 111

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Abstract

We consider the problem of on-line searching on m rays. A point robot is assumed to stand at the origin of m concurrent rays one of which contains a goal g that the point robot has to reach. Neither the ray containing g nor the distance to g are known to the robot. The only way the robot can detect g is by reaching its location. We use the *competitive ratio* to measure the performance of a strategy.

We present a simple proof of a tight lower bound for randomized strategies to search on m rays. In particular, we show in our proof that there is an optimal randomized strategy that visits the rays in cyclic order. We also obtain a bound on the convergence rate to the optimal competitive ratio for a fixed m if the goal has a distance of at most D to the origin.

Finally, we show that the optimal competitive ratio is asymptotically given by $1 + 2 \cdot 1.544\, m$ as m goes to infinity.

1 Introduction

Searching for a goal is an important and well studied problem in robotics. In many realistic situations the robot does not possess complete knowledge about its environment, for instance, the robot may not have a map of its surroundings or the location of the goal may be unknown [BRS93, CL93, DHS95, DI94, IK95, Kle92, Kle94, LOS95, MI94, PY89].

Since the robot has to make decisions about the search based only on the part of its environment that it has explored before, the search of the robot can be viewed as an *on-line* problem. This invites the application of *competitive analysis* to judge the performance of an on-line search strategy. The *competitive ratio* of a search strategy is defined as the worst case ratio of the distance traveled by the robot to the distance from the origin to the goal, maximized over all possible locations of the goal [ST85].

Searching on m concurrent rays is a problem with paradigmatic status in this setting. Here, a point robot is imagined to stand at the origin s of m rays one of which contains the

*This research is supported by the DFG-Project "Diskrete Probleme", No. Ot 64/8-2.

goal g whose distance to the origin is unknown. The robot can only detect g if it stands on top of it. Many problems of searching in more complex geometries can be reduced to searching on m rays [DI94, LOS97, LOS96, DHS95].

In the deterministic case Baeza-Yates *et al.* [BYCR93] and Gal [Gal80] present an optimal search strategy that achieves a competitive ratio of $1 + 2m^m/(m-1)^{m-1}$. In the randomized case Kao, Reif, and Tate [KRT93] present an optimal randomized algorithm to search on m rays and prove its optimality in the special case $m = 2$ (see also [Gal80]). The algorithm achieves a competitive ratio of

$$c_m^* = 1 + \frac{2}{m} \min_{a>1} \frac{a^m - 1}{(a-1)\ln a}.$$

For general m, Kao *et al.* prove a matching lower bound [KMSY94]. It should be noted, however, that the proof by Kao *et al.* only applies to *periodic* strategies, that is, strategies that visit the rays in a fixed cyclic order [KMSY94]; a proof of the optimality of periodic strategies is given by Ma and Yin [MY94] but remains unpublished to date. In this paper we partially reprove and extend the results by Kao *et al.* and Ma and Yin using a different and simpler approach. More precisely, the contribution of this paper is threefold.

1. We present a tight lower bound on the competitive ratio of randomized strategies to search on m rays that, in particular, shows that there is an optimal strategy that is periodic. In addition, our proof is simpler and more elementary than that by Kao *et al.* [KMSY94].

2. The lower bound we prove also applies to searching on bounded rays; that is, our proof provides a lower bound even if we assume that a search algorithm is given an upper bound on the maximal distance to the target in advance.

3. In order to estimate the competitive ratio of the optimal randomized strategy for large m it is useful to know its constant of proportionality. We show that the competitive ratio c_m^* of randomized searching on m rays is $\approx 1 + 2 \cdot 1.544\, m$, as m goes to infinity.

The paper is organized as follows. In the next section we introduce some definitions and present the optimal randomized algorithm to search on m rays. In Section 3 we prove a lower bound for randomized strategies for searching on m rays. In particular, we prove the optimality of periodic strategies and show how to adapt our lower bound proof to the case when an upper bound D on the distance to target is known. Finally, in Section 4 we show that c_m^* approaches $1 + 2 \cdot 1.544\, m$ as m goes to infinity.

2 Randomized m-Way Ray Searching

Consider a randomized search strategy X to search on m rays. The distance $dist_X(g)$ traveled by a robot using X to find the goal g depends on the random choices of X and is a random variable. We define the competitive ratio of X to be the ratio of the expected distance of $dist_X(g)$ over the distance to the goal. In other words, the strategy X has competitive ratio c_X if, for all goal positions g,

$$E[dist_X(g)] \leq c_X dist(g). \tag{1}$$

Sometimes an additive constant on the right hand side of Equation 1 is allowed to avoid degeneracies if the goal is very close. We choose a different solution by assuming that the distance to the goal is at least one.

2.1 An Optimal Algorithm

The following algorithm to search on m rays is presented and analysed by Kao *et al.* [KRT93, KMSY94]. A slight variant of it was first suggested by Gal [Gal80].

Algorithm Randomized m-way Ray-Search
let σ be a random permutation of $\{0, \ldots, m-1\}$
let a_m^* be the number that minimizes the function $a^m/((a-1)\ln a)$
let ε be a random real uniformly chosen from $[0, 1)$
let $d_0 = (a_m^*)^\varepsilon$; $i = 0$
while the target is not found **do**
  let $r := \sigma(i \bmod m)$
  explore ray r up to distance d_i
  $d_{i+1} := d_i \cdot a_m^*$
  $i := i + 1$
end while

It can be shown that Algorithm *Randomized m-way Ray-Search* has a competitive ratio of

$$c_m^* = 1 + \frac{2}{m}\frac{(a_m^*)^m}{(a^*-1)\ln a^*}.$$

Note that Algorithm *Randomized m-way Ray-Search* is an example of a *periodic* strategy, that is, for every $i \geq 0$, the ray visited in Step $i+m$ is the same ray as visited in Step i. In next section we prove a matching lower bound for the competitive ratio of any randomized search strategy.

3 Proving a Lower Bound

In order to prove the lower bound for randomized searching on m-rays we make use of Yao's corollary to the well-known von Neumann minimax principle [Yao77]. It states that a lower bound for *deterministic* strategies on a given input distribution is also a lower bound for a randomized strategy on its worst case input.

So in the following we deal with deterministic strategies and probability distributions on m rays which we call *hiding strategies*. We show that, for every $\varepsilon > 0$, there is a hiding strategy H, such that, for each deterministic search strategy X, the expected competitive ratio $E_H(X)$ of X under H is at least $(1-\varepsilon)^3 c_m^*$. By Yao's lemma this implies that the competitive ratio of a randomized strategy X on its worst case input is at least c_m^*.

3.1 Search and Hiding Strategies

We represent a deterministic search strategy as an infinite sequence X of pairs (x_k, J_k) where x_k is the distance that the robot travels in Step k along a given ray until it returns and J_k is the index of the next time when the robot travels along the same ray again [Gal80]. Note that this representation is unique up to an ordering of the sequence in which the rays are visited the first time. Since in our case the rays are indistinguishable in the beginning, we leave the sequence in which the rays are visited the first time unspecified. We define $J^{-1}(k)$ to be the inverse of the function J; that is, if there is an index i with $J_i = k$, then we define $J^{-1}(k) = i$; if the robots visits the ray for the first time in Step k, then we define $x_{J^{-1}(k)} = 0$. If X is a periodic strategy, then $J_k = k + m$ and $J^{-1}(k) = k - m$.

We model a hiding strategy H as a collection of m measures $H_r : [0, \infty) \longrightarrow I\!R^+$, $0 \le r \le m-1$, with

$$\sum_{r=0}^{m-1} \int_0^\infty dH_r(h) = 1.$$

where $\int_0^{h_0} dH_r(h)$ is the probability that the target is hidden in ray r up to a distance of at most h_0. Hence, the additional contribution to the expected competitive ratio when the robot explores the ray r_k in Step k up to distance x_k is

$$\int_{x_{J^{-1}(k)}}^{x_k} \left(2 \sum_{i=0}^{k-1} x_i + h \right) \frac{dH_{r_k}(h)}{h}$$

since ray r_k has been explored before up to distance $x_{J^{-1}(k)}$ and the robot has traveled a distance of $2 \sum_{i=0}^{k-1} x_k$ before Step k. Therefore, the expected competitive ratio $E_H(X)$ of search strategy X is given by

$$E_H(X) = \sum_{k=0}^{\infty} \int_{x_{J^{-1}(k)}}^{x_k} \left(2 \sum_{i=0}^{k-1} x_i + h \right) \frac{dH_{r_k}(h)}{h} = 1 + 2 \sum_{k=0}^{\infty} \sum_{i=0}^{k-1} x_i \int_{x_{J^{-1}(k)}}^{x_k} \frac{dH_{r_k}(h)}{h}. \qquad (2)$$

3.2 Min-Increasingness

We call a hiding strategy H *uniform* if the hiding strategy H_r for ray r equals the hiding strategy $H_{r'}$ for ray r', for all rays r and r'. In the following we prove that, for all uniform hiding strategies, there is an optimal deterministic search strategy that is min-increasing. A strategy X is called *min-increasing* if the robot always explores a ray r in Step k that has been explored the least so far; in other words, if d_r is the distance up to which ray r is explored before Step k by strategy X, then the robot chooses a ray in Step k which minimizes d_r. Since the explored distance d_{r_k} of the current ray r_k equals $x_{J^{-1}(k)}$, this implies that $x_{J^{-1}(k)} \le x_{J^{-1}(k+1)}$, for all $0 \le k < \infty$.

In the following we show that all optimal deterministic search strategies for uniform hiding strategies on m rays are min-increasing.

Lemma 3.1 *Let H be a uniform hiding strategy and let X be a deterministic strategy to search on m rays. If X is not min-increasing, then there is a strategy X' such that*

$$E_H(X) > E_H(X').$$

Proof: Let X be a non-min-increasing strategy and $n > 0$ be an index with $x_{J^{-1}(n)} > x_{J^{-1}(n+1)}$. We now define strategy $X' = (x_n, J'_n)$ as follows.

$$J'_k = \begin{cases} J_k & \text{if } k \ne J^{-1}(n), J^{-1}(n+1) \\ n+1 & \text{if } k = J^{-1}(n) \\ n & \text{if } k = J^{-1}(n+1). \end{cases}$$

Intuitively, in Strategy X' we explore the ray r_{n+1} in Step n and then the ray r_n in Step $n+1$ and at the same time we switch the roles of r_n and r_{n+1}. After exchanging r_n and r_{n+1} the situation for X and X' is identical. The sequence on ray r_n of return points after Step n in X is $x_n, x_{J_n}, \ldots$, which is the same sequence as for ray r_{n+1} after Step n in X'. Note that this implies that the contribution of x_n and x_{n+1} to the expected cost after Step $n+1$ is the same for X and X'. Clearly, the situation is also indistinguishable before Step n. Therefore, there is only a difference in Steps n and $n+1$. More precisely, if we take into account that $H_r = H_0$, for all $0 \le r \le m-1$, then

$$\begin{aligned} E_H(X) - E_H(X') &= 2\sum_{k=0}^{\infty}\sum_{i=0}^{k-1} x_i \int_{x_{J^{-1}(k)}}^{x_k} \frac{dH_0(h)}{h} - \int_{x_{J^{-1}(k)'}}^{x_k} \frac{dH_0(h)}{h} \\ &= 2\sum_{k=n}^{n+1}\sum_{i=0}^{k-1} x_i \int_{x_{J^{-1}(k)}}^{x_k} \frac{dH_0(h)}{h} - \int_{x_{J^{-1}(k)'}}^{x_k} \frac{dH_0(h)}{h} \\ &= 2\sum_{i=0}^{n-1} x_i \int_{x_{J^{-1}(n)}}^{x_{J^{-1}(n+1)}} \frac{dH_0(h)}{h} + 2\sum_{i=0}^{n} x_i \int_{x_{J^{-1}(n+1)}}^{x_{J^{-1}(n)}} \frac{dH_0(h)}{h} \\ &= 2x_n \int_{x_{J^{-1}(n+1)}}^{x_{J^{-1}(n)}} \frac{dH_0(h)}{h} > 0 \end{aligned}$$

since $x_{J^{-1}(n)} > x_{J^{-1}(n+1)}$. □

3.3 Periodicity of Optimal Strategies

Let H be a uniform hiding strategy. In the following we show that there is an optimal deterministic strategy to search on m rays that visits the rays in cyclic order.

Let X be an optimal deterministic strategy to search for a goal on m rays under the uniform hiding strategy H. We define a new sequence $X^s = (x_0^s, x_1^s, x_2^s, \ldots)$ by $x_k^s = x_{J^{-1}(k)}$, for $k \ge 0$. We set $x_k^s = 0$ for the Steps i in which a ray is visited the first time. Since strategy X is min-increasing by Lemma 3.1, the sequence X^s is non-decreasing. It can be viewed as the sorted sequence X. By Equation 2 the expected cost of Strategy X is given by

$$E_H(X) = 1 + 2\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k-1} x_i\right)\int_{x_{J^{-1}(k)}}^{x_k} \frac{dH_{r_k}(h)}{h} = 1 + 2\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k-1} x_{J_i}^s\right)\int_{x_k^s}^{x_{J_k}^s} \frac{dH_{r_k}(h)}{h}.$$

We claim that

$$\sum_{i=0}^{k-1} x_{J_i}^s \ge \sum_{i=m}^{k+m-1} x_i^s.$$

To see the above inequality we first observe that because Strategy X is min-increasing, all the step lengths that are smaller than $x_k^s = x_{J^{-1}(k)}$ have been explored before Step k. Hence, all the values x_i^s, with $m \le i \le k-1$, are contained in $\sum_{i=0}^{k-1} x_{J_i}^s$. Secondly, consider the $m-1$ rays that are not explored in Step k. Again since Strategy X is min-increasing, all of these rays are explored up to a distance of at least x_k^s. This implies that the sum $\sum_{i=0}^{k-1} x_{J_i}^s$ contains $m-1$ values that are at least as large as x_k^s. The smallest possible choice for these values is $x_{k+1}^s, x_{k+2}^s, \ldots, x_{k+m-1}^s$.

With the above inequality the expected cost for X can be bounded from below by

$$E_H(X) \;\geq\; 1 + 2\sum_{k=0}^{\infty}\sum_{i=m}^{k+m-1} x_i^s \int_{x_k^s}^{x_{J_k}^s} \frac{dH_{r_k}(h)}{h} \;\geq\; 1 + 2\sum_{k=m}^{\infty} x_k^s \sum_{r=0}^{m-1} \int_{x_{k-r}^s}^{\infty} \frac{dH_r(h)}{h}. \quad (3)$$

The second inequality in (3) is shown as follows. Consider the value x_k^s and a ray r. Since the integrals are additive, we only need to consider the first step i in which ray r is explored such that x_k^s appears in the sum $\sum_{j=m}^{i+m-1} x_j^s$. Since the ray that is explored up to distance x_k^s is not chosen in Step i, the min-increasingness of Strategy X implies that $x_k^s \geq x_i^s$ and x_k^s is one of the $m-1$ values $x_{i+1}^s, \ldots, x_{i+m-1}^s$. Since x_k^s is a different of these values for each of the $m-1$ other rays, the integration for x_k^s starts at the values $x_{k-m+1}^s, \ldots, x_{k-1}^s$ and the total contribution of x_k^s to the expected cost is

$$x_k^s \sum_{r=0}^{m-1} \int_{x_{k-r}^s}^{\infty} \frac{dH_r(h)}{h}$$

as claimed. If we now shift the indices of X^s by m back and define $x_{-m} = x_{-m+1} = \cdots = x_{-1} = 0$, then we obtain that

$$E_H(X) \;\geq\; 1 + 2\sum_{k=0}^{\infty} x_k^s \sum_{r=0}^{m-1} \int_{x_{k-r}^s}^{\infty} \frac{dH_r(h)}{h} \quad (4)$$

where X^s is a non-decreasing positive sequence. The right hand side of the above inequality can be viewed as the expected cost of X^s if we explore the m rays in a cyclic order. Hence, we have shown the following theorem.

Theorem 3.2 *If H is a uniform hiding strategy, then there is an optimal deterministic strategy to search on m rays that is periodic and non-decreasing.*

In combination with Yao's Lemma Theorem 3.2 implies that there is an optimal periodic and non-decreasing randomized search strategy.

3.4 The Hiding Strategy H_D

In order to prove a lower bound for searching on m rays we need to define a specific hiding strategy. We use a generalization of the hiding strategy used by Gal for the case $m = 2$ [Gal80]. Let $D > 0$. The uniform hiding strategy H_D is now defined by the measure

$$H_D(h) = \begin{cases} 0 & \text{for } 0 \leq h \leq 1 \\ \dfrac{\ln h}{m(1+\ln D)} & \text{for } 1 \leq h \leq D \\ \dfrac{1}{m(1+\ln D)} & \text{for } h \geq D, \end{cases}$$

for each of the m rays.[1] Let $\delta_D = 1/(1+\ln D)$. The probability measure H_D has a density of $\delta_D/(m\,h)$, for $1 \leq h < D$, and a probability atom of mass δ_D/m at $h = D$. The hiding

[1] The hiding strategy H used by Kao *et al.* [KRT93, KMSY94] is given by $H(h) = -h^{-\varepsilon}/m$, for $h \geq 1$. This leads to a much more complicated optimization problem than the problem $(*)$ below. Since H is uniform, Theorem 3.2 also applies H.

strategy H_D has the following very useful property for $1 \le x \le D$.

$$\int_x^\infty \frac{dH_D(h)}{h} = \int_x^D \frac{dH_D(h)}{h} = \frac{\delta_D}{m}\int_x^D \frac{1}{h^2}dh + \frac{\delta_D}{m}\frac{1}{D} = \frac{\delta_D}{m}\left(\frac{1}{x} - \frac{1}{D} + \frac{1}{D}\right) = \frac{1}{m(1+\ln D)}\frac{1}{x}.$$

Let $\varepsilon > 0$ be arbitrary. We show the following lemma.

Lemma 3.3 *If X is a strategy with $E_{H_D}(X) \le c_m^*$, then, for every $\varepsilon > 0$, there is a $D > 0$ such that*

$$E_{H_D}(X) \ge (1-\varepsilon)^3 \left(1 + 2\frac{(a_m^*)^m - 1}{m(a_m^* - 1)\ln a_m^*}\right).$$

The remainder of this section is devoted to the proof of Lemma 3.3.

3.5 A Minimization Problem

In the following let X be an optimal, periodic, non-decreasing, deterministic strategy to search on m rays under the hiding function H_D. Let n be the first index such that $x_n \ge D$. The expected cost of X is given by

$$\begin{aligned} E_{H_D}(X) &= 1 + 2\sum_{k=0}^{\infty} x_k \sum_{r=0}^{m-1}\int_{x_{k-r}}^{\infty}\frac{dH_r(h)}{h} = 1 + 2\sum_{k=0}^{n+m-1} x_k \sum_{r=0}^{m-1}\int_{x_{k-r}}^{D}\frac{dH_r(h)}{h} \\ &\ge 1 + 2\frac{1}{m(1+\ln D)}\left(\sum_{k=0}^{n}\sum_{r=0}^{m-1}\frac{x_k}{x_{k-r}} - 1\right) \qquad (5) \\ &\ge 1 + 2\frac{\sum_{k=0}^{n}\sum_{r=0}^{m-1} x_k/x_{k-r} - 1}{m(1+\ln x_n)} \end{aligned}$$

where we define $x_{-m+1} = \cdots = x_{-1} = 1$. Note that we need to subtract 1 in Equation 5 as x_n is not integrated along the ray $r_{n \bmod m}$ with a positive measure anymore. Hence, we are interested in a lower bound for the ratio

$$\frac{-1 + \sum_{k=0}^{n}\sum_{r=0}^{m-1} x_k/x_{k-r}}{m(1+\ln x_n)}. \qquad (6)$$

Let $a_i = x_i/x_{i-1}$, for $-m+2 \le i \le n$. Hence, $x_n = \prod_{i=-m+2}^{n} a_i$. Since $a_i = 1$, for $i < 0$, we have $x_n = \prod_{i=0}^{n} a_i$. Hence,

$$E_{H_D}(X) \ge \frac{\sum_{k=0}^{n}\sum_{r=0}^{m-1} x_k/x_{k-r} - 1}{m(1+\ln x_n)} = \frac{\sum_{k=0}^{n}\sum_{r=0}^{m-1}\prod_{i=k-r+1}^{k} a_i - 1}{m(1+\ln \prod_{i=0}^{n} a_i)},$$

where we define $\prod_{i=k-r+1}^{k} a_i = 1$, if $i - r + 1 > i$.

If we assume that $x_n = \prod_{i=0}^{n} a_i$ is constant, then we want to solve the following minimization problem

$$(*) \quad \text{minimize} \quad -1 + \sum_{k=0}^{n}\left(1 + a_k + a_k a_{k-1} + \cdots + \prod_{i=k-m+2}^{k} a_i\right)$$

$$\text{where} \quad \prod_{i=0}^{n} a_i = x_n \quad \text{and} \quad a_{-m+1} = \cdots = a_{-1} = 1 \qquad (7)$$

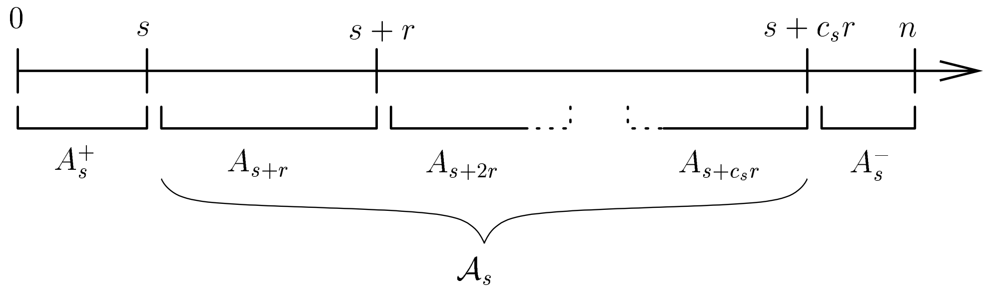

Figure 1: The products that belong to set $\mathcal{A}_s$.

Note that the objective function as well as the set of admissible solutions of the above minimization problem are highly non-convex. Instead of solving the above problem directly we minimize each of the $m-1$ sums $S_r = \sum_{k=0}^{n} \prod_{i=k-r+1}^{k} a_i$, for $1 \le r \le m-1$, separately, under the constraint (7). The sum of the $m-1$ lower bounds we obtain in this way is obviously a lower bound for Expression 6.

3.6 Minimizing the Sum S_r

In the following let α_n be the $(n+1)$st root of x_n and let r be a fixed integer between 1 and $m-1$. The main idea of the proof is to show that the product A_k of the r numbers $a_{k-r+1}, \ldots, a_k$ equals approximately the r-th power of α_n; that is, on the average a_k is about α_n.

We split the terms of the sum $S_r = \sum_{k=0}^{n} A_k$ into $r+1$ sets $\mathcal{A}_{-1}, \mathcal{A}_0, \ldots, \mathcal{A}_{r-1}$ where the product A_k belongs to set $\mathcal{A}_s$ if $k \bmod r = s$ and $k \ge r-1$. The remaining products A_k belong to $\mathcal{A}_{-1}$ (see Figure 1). In the following we show that the products in the sets $\mathcal{A}_0, \ldots, \mathcal{A}_{r-1}$ have all the same value.

Lemma 3.4 *If $A_k \ne A_{k+r}$, for some $r-1 \le k \le n-r$, then there is a sequence $(\overline{a}_{-m+1}, \overline{a}_0, \ldots, \overline{a}_n)$ that satisfies the Constraints 7 and, additionally,*

$$\sum_{k=0}^{n} \overline{A}_k < \sum_{k=0}^{n} A_k$$

where $\overline{A}_k = \prod_{i=k-r+1}^{k} \overline{a}_i$, for $r-1 \le k \le n$.

Proof: Let $r-1 \le k_0 \le n-r$ with $A_{k_0} \ne A_{k_0+r}$, say, $A_{k_0} < A_{k_0+r}$. Then, there is a $\alpha > 0$ such that $A_{k_0}(1+\alpha) = A_{k_0+r}/(1+\alpha)$. If we define a new sequence by

$$\overline{a}_i = \begin{cases} a_i & \text{if } i \ne k_0, \\ (1+\alpha)a_i & \text{if } i = k_0, \text{ and} \\ a_i/(1+\alpha) & \text{if } i = k_0+1, \end{cases}$$

then

$$\sum_{k=0}^{n} \prod_{i=k-r+1}^{k} \overline{a}_i \quad = \quad \sum_{\substack{k=0 \\ k \ne k_0, k_0+r}}^{n} A_k + \frac{A_{k_0+r}}{1+\alpha} + (1+\alpha)A_{k_0} < \sum_{k=0}^{n} A_k,$$

where the last inequality follows from the fact that $x/(1+\alpha)+y(1+\alpha)$ achieves its minimum for $x/(1+\alpha)=y(1+\alpha)$. Hence, we can replace a_{k_0} and a_{k_0+1} by $\overline{a}_{k_0}$ and $\overline{a}_{k_0+1}$ and obtain a squence where the product of the elements is still x_n but the sum of the products A_k is smaller. □

By a simple induction it can now be seen that the products in each $\mathcal{A}_s$ have the same value. If $s=1$, then we have the following corollary of Lemma 3.4.

Corollary 3.5

$$\sum_{k=0}^{n} a_k \geq (n+1)\alpha_n.$$

Proof: By Lemma 3.4 $\sum_{k=0}^{n} a_k$ is minimized under Constraint 7 if $a_0=a_1=\cdots=a_n$. Since $\prod_{k=0}^{n} a_k = x_n$, we obtain $a_k=\alpha_n$, for $0\leq k\leq n$. □

Let c_s be the cardinality of $\mathcal{A}_s$. If we consider the product $\prod_{k\in\mathcal{A}_s} A_k$, then this product contains almost all a_k (except for the at most m values $a_0,\ldots,a_s$ at the beginning and the at most m values $a_{s+c_s r+1},\ldots,a_n$ at the end). In the following we show that the equality of the values of A_k, for $k\in\mathcal{A}_s$, implies that A_k converges to α_n^r as D goes to infinity.

In order to do this we need to bound the product of the remaining at most $2m$ values a_i that are not part of the product $\prod_{i\in\mathcal{A}_s} A_k$. We first give a lower bound on the size of n and an upper bound on the size of x_n and of a_i, $0\leq i\leq n$. In the following let γ_m^* be the unique solution of the equation $c/(1+\ln c)=mc_m^*$.[2]

Lemma 3.6 *If X is a strategy with $E_{H_D}(X)\leq c_m^*$, then*

1. $x_n \leq (\gamma_m^*)^{n+1}$,
2. $n+1\geq \ln D/\ln\gamma_m^*$, *and*
3. $a_i\leq mc_m^*(1+\ln D)$, *for* $0\leq i\leq n$.

Proof: We first prove that $x_n\leq(\gamma_m^*)^{n+1}$. We have

$$\begin{aligned}\frac{\gamma_m^*}{m(1+\ln\gamma_m^*)} &= c_m^* \;\geq\; E_{H_D}(X) \;\geq\; \frac{\sum_{i=0}^{n} a_i}{m(1+\ln x_n)} \overset{\text{(Corollary 3.5)}}{\geq} \frac{(n+1)\alpha_n}{m(1+(n+1)\ln\alpha_n)}\\ &= \frac{\alpha_n}{m(1/(n+1)+\ln\alpha_n)} \;\geq\; \frac{\alpha_n}{m(1+\ln\alpha_n)}\end{aligned}$$

Therefore, $\alpha_n\leq\gamma_m^*$ and $x_n\leq(\gamma_m^*)^{n+1}$.

The second inequality follows from the first by $D\leq x_n\leq(\gamma_m^*)^{n+1}$, that is, $\ln D\leq(n+1)\ln\gamma_m^*$. Finally, we have by Equation 5

$$c_m^*\geq E_{H_D}(X) \;\geq\; \frac{a_i}{m(1+\ln D)}$$

and the third claim follows. □

We now can give a lower bound on S_r.

[2]Note that γ_m^* exists since $x/(1+\ln x)$ is a monotone function for $x\geq 1$.

Lemma 3.7 *If D is large enough, then*

$$S_r \geq (1-\varepsilon)(n-m)\alpha_n^r$$

Proof: Let $0 \leq s \leq r-1$. Since $A_k = A_{k'}$, for all $k, k' \in \mathcal{A}_s$ by Lemma 3.4, we can define $A_s^* \stackrel{def}{=} A_k$, with $k \in \mathcal{A}_s$, and A_s^* only depends on s and not on k. Recall that c_s is the size of $\mathcal{A}_s$. The sum S_r is now given by

$$S_r \;=\; \sum_{k=0}^{n} A_k \;\geq\; \sum_{s=1}^{r}\sum_{k\in\mathcal{A}_s} A_k \;=\; \sum_{s=1}^{r} c_s A_s^*. \tag{8}$$

Let $A_s^+ = a_{-m+1}\cdots a_s$, and $A_s^- = a_{s+c_s r+1}\cdots a_n$ (see Figure 1). We have, for $0 \leq s \leq r-1$,

$$A_s^+ (A_s^*)^{c_s} A_s^- \;=\; A_s^+ \prod_{k=1}^{c_s}\left(\prod_{i=s+(k-1)r+1}^{s+kr} a_i\right) A_s^- = A_s^+\left(\prod_{i=s+1}^{s+c_s r} a_i\right) A_s^- = x_n,$$

Hence,

$$A_s^* = \frac{x_n^{1/c_s}}{(A_s^+ A_s^-)^{1/c_s}} = \frac{\alpha_n^{(n+1)/c_s}}{(A_s^+ A_s^-)^{1/c_s}} \geq \frac{\alpha_n^{(n+1-c_s r+c_s r)/c_s}}{(A_s^+ A_s^-)^{(m+1)/n}} \stackrel{(n+1\geq c_s r)}{\geq} \frac{\alpha_n^r}{(A_s^+ A_s^-)^{(m+1)/n}} \tag{9}$$

By Lemma 3.6

$$\begin{aligned} A_s^+ &= a_{-m+1}\cdots a_s & &\leq (mc_m^*(1+\ln D))^s & &\leq (mc_m^*(1+\ln D))^{m-1} \quad \text{and} \\ A_s^- &= a_{s+c_s r+1}\cdots a_n & &\leq (mc_m^*(1+\ln D))^r & &\leq (mc_m^*(1+\ln D))^{m-1}. \end{aligned}$$

Since n grows with $\ln D$ by Lemma 3.6, we can choose D large enough such that

$$\frac{1}{(A_s^+ A_s^-)^{(m+1)/n}} \;\geq\; \frac{1}{(mc_m^*(1+\ln D))^{(m^2-1)/n}} \geq (1-\varepsilon). \tag{10}$$

Combining the above results with Equation 8 we obtain

$$\begin{aligned} S_r = \sum_{k=0}^{n} A_k \;&\geq\; \sum_{s=1}^{r} c_s A_s^* \;\stackrel{(9)}{\geq}\; \sum_{s=1}^{r} c_s \frac{\alpha_n^r}{(A_s^+ A_s^-)^{(m+1)/n}} \;\stackrel{(10)}{\geq}\; (1-\varepsilon)\alpha_n^r \sum_{s=1}^{r} c_s \\ &\geq\; (1-\varepsilon)\,\alpha_n^r\,(n-m) \end{aligned}$$

as claimed. □

3.7 Establishing the Lower Bound

We now make use of the lower bounds for the sums S_r that we have obtained above.

$$\begin{aligned} \frac{-1+\sum_{k=0}^{n}\sum_{r=0}^{m-1}\prod_{i=k-r+1}^{k} a_j}{m(1+\ln\prod_{i=0}^{n} a_i)} \;&\geq\; \frac{\sum_{r=0}^{m-1}(1-\varepsilon)\alpha_n^r(n-m)}{m(1+(n+1)\ln\alpha_n)} - \frac{1}{m(1+\ln D)} \\ &=\; \frac{1-\varepsilon}{m}\frac{n-m}{n+1}\frac{\sum_{r=0}^{m-1}\alpha_n^r}{\ln\alpha_n + 1/(n+1)} - \frac{1}{m(1+\ln D)} \\ \text{(see below)}\;&\geq\; \frac{(1-\varepsilon)^2}{m(1+\varepsilon)}\frac{\sum_{r=0}^{m-1}\alpha_n^r}{\ln\alpha_n} - \frac{1}{m(1+\ln D)} \\ &\geq\; \frac{(1-\varepsilon)^3}{m}\frac{\sum_{r=0}^{m-1}\alpha_n^r}{\ln\alpha_n} - \frac{\varepsilon}{m}. \end{aligned} \tag{11}$$

Here, Inequality 11 can be seen as follows. First we choose D large enough such that $(n-m)/(n+1) \le (1-\varepsilon)$ and $\ln D \ge 1/\varepsilon$. We obtain

$$(1+\varepsilon)\ln \alpha_n \quad \overset{x_n > D}{\ge} \quad \ln \alpha_n + \varepsilon \ln D^{1/(n+1)} \quad \ge \quad \ln \alpha_n + \frac{1}{n+1}$$

which proves the inequality. Thus, we have proven that, for all $\varepsilon > 0$, there exists a $D > 0$ such that, for all periodic deterministic strategies X that search m rays

$$E_{H_D}(X) \quad \ge \quad (1-\varepsilon) + (1-\varepsilon)^3 \frac{2}{m} \min_{1<a<\infty} \frac{\sum_{k=0}^{m-1} a^k}{\ln a}.$$

This completes the proof of Lemma 3.3

Let $E_g(X)$ denote competitive ratio of strategy X if the goal is placed at position g. By Yao's lemma the above result implies the following theorem.

Theorem 3.8 *For every $\varepsilon > 0$ and for every randomized strategy X that searches on m rays, there is a goal position g such that*

$$E_g(X) \ge (1-\varepsilon) + (1-\varepsilon)^3 \frac{2}{m} \frac{\sum_{k=0}^{m-1} (a_m^*)^k}{\ln a_m^*}.$$

Theorem 3.8 and implies the following corollary as already shown by Kao *et al.* [KMSY94].

Corollary 3.9 *The Strategy* Randomized m-way Ray-Search *is an optimal randomized search strategy.*

3.8 Bounded Searching

In fact, our analysis also yields a bound on the convergence rate for optimal randomized strategies if an upper bound on the distance to the goal is known.

Theorem 3.10 *Let X be an optimal randomized search strategy to search m rays. If the maximum distance of the origin to the goal is bounded by D, then there is a goal position g such that*

$$E_g(X) \quad \ge \quad \left(1 - O\left(\frac{m^2 \ln m \ln \ln D}{\ln D}\right)\right)\left(1 + 2\frac{\sum_{k=0}^{m-1}(a_m^*)^k}{m \ln a_m^*}\right), \tag{12}$$

for large enough D.

Proof: Assume that we are given an upper bound D on the distance to the goal. We again make use of Yao's lemma. Hence, we have to provide a hiding strategy H that is bounded by D such that any deterministic strategy to search on m rays has a competitve ratio as in Equation 12.

Of course, the hiding strategy H_D is well suited. Since we assumed above that H_D is known to the strategy, all the bounds we have shown remain valid.

So let X be an optimal deterministic strategy to search on m rays if we are given H_D. By our previous analysis we have

$$E_{H_D}(X) \geq 1 + \frac{1}{(mc_m^*(1+\ln D))^{(m^2-1)/n}} \frac{n-m}{n+1} \frac{1}{1+\frac{1}{\ln D}} \frac{2\sum_{r=0}^{m-1}\alpha_n^r}{m\ln\alpha_n} - \frac{1}{m(1+\ln D)}$$

since

$$\left(1+\frac{1}{\ln D}\right)\ln\alpha_n = \ln\alpha_n + \frac{\ln\alpha_n^{n+1}}{\ln D}\frac{1}{n+1} = \ln\alpha_n + \frac{\ln x_n}{\ln D}\frac{1}{n+1} \geq \ln\alpha_n + \frac{1}{n+1}.$$

Since $n+1 \geq \ln D/\ln\gamma_m^*$ by Lemma 3.6, we obtain

$$\begin{aligned}
E_{H_D}(X) &\geq 1 + \frac{1}{(mc_m^*(1+\ln D))^{(m^2-1)/n}} \frac{\ln D/\ln\gamma_m^* - m - 1}{\ln D/\ln\gamma_m^*} \\
&\quad \left(1-\frac{1}{\ln D}\right)\frac{2\sum_{r=0}^{m-1}\alpha_n^r}{\ln\alpha_n} - \frac{1}{m(1+\ln D)} \\
&= 1 + \frac{1}{(mc_m^*(1+\ln D))^{(m^2-1)/n}} \left(1-\frac{(m+1)\ln\gamma_m^*}{\ln D}\right) \\
&\quad \left(1-\frac{1}{\ln D}\right)\frac{2\sum_{r=0}^{m-1}\alpha_n^r}{\ln\alpha_n} - \frac{1}{m(1+\ln D)}.
\end{aligned}$$

Now, if D is large enough, then

$$\begin{aligned}
\ln(mc_m^*(1+\ln D))^{(m^2-1)/n} &\leq \frac{(m^2-1)}{\ln D/\ln\gamma_m^* - 1}\ln(mc_m^*(1+\ln D)) \\
&= \frac{(m^2-1)\ln\gamma_m^*\ln(mc_m^*(1+\ln D))}{\ln D - \ln\gamma_m^*} \\
&\leq 2\ln\left(1+\frac{(m^2-1)\ln\gamma_m^*\ln(mc_m^*(1+\ln D))}{\ln D - \ln\gamma_m^*}\right)
\end{aligned}$$

and, thus,

$$\begin{aligned}
\frac{1}{(mc_m^*(1+\ln D))^{(m^2-1)/n}} &\geq \left(1-\frac{(m^2-1)\ln\gamma_m^*\ln(mc_m^*(1+\ln D))}{\ln D - \ln\gamma_m^*}\right)^2 \\
&\geq \left(1-\frac{2(m^2-1)\ln\gamma_m^*\ln(mc_m^*(1+\ln D))}{\ln D - \ln\gamma_m^*}\right).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\left(1-\frac{2(m^2-1)\ln\gamma_m^*\ln(mc_m^*(1+\ln D))}{\ln D - \ln\gamma_m^*}\right)\left(1-\frac{(m+1)\ln\gamma_m^*}{\ln D}\right)\left(1-\frac{1}{\ln D}\right) \\
&\qquad \geq \left(1-\frac{2(m+1)\ln\gamma_m^*(1+(m-1)(\ln(1+\ln D)+\ln mc_m^*))+1}{\ln D - \ln\gamma_m^*}\right)
\end{aligned}$$

and with $c_m^* = O(m)$ and $\gamma_m^* = O(m^2\log m)$ the claim follows. □

m	Opt. rand. step length a_m^*	α_m	Rand. comp. ratio c_m^*	$\frac{c_m^*-1}{2m}$	Opt. det. step length	Det. comp. ratio d_m	$\frac{c_m^*-1}{c_m^*-1}$
2	3.591	5.182	4.591	0.898	2.0	9.0	0.449
3	2.011	3.033	7.732	1.122	1.5	14.5	0.499
4	1.622	2.488	10.841	1.23	1.333	19.963	0.519
5	1.448	2.24	13.942	1.294	1.25	25.414	0.53
6	1.350	2.1	17.037	1.336	1.167	30.860	0.537
7	1.287	2.009	20.130	1.366	1.143	36.302	0.542
10	1.186	1.866	29.404	1.42	1.111	52.623	0.55
20	1.086	1.719	60.296	1.482	1.053	107.001	0.559
50	1.033	1.642	152.95	1.519	1.02	270.105	0.565
100	1.016	1.617	307.365	1.532	1.01	541.936	0.566

Table 1: A comparison of the optimal randomized and deterministic strategy to search on m rays.

4 The Asymptotic Competitive Ratio

In this section we investigate the growth rate of the competitive ratio for randomized search strategies on m rays. As Table 1 shows it seems that the competitive ratio grows linearly with the number of rays. In the following we are going to compute the exact constant of proportionality. Knowing this constant allows us to make fairly accurate estimates for the competitive ratio of randomized searching for arbitrary values of m. It also provides a basis for assessing the maximal gain of a randomized strategy over the best deterministic strategy. As a side result we also obtain an estimate for the step length.

In the deterministic case it is easy to see that the constant of proportionality is $2e$ since we have an explicit solution. The competitive ratio is given by

$$1+2\frac{m^m}{(m-1)^{m-1}} = 1+2m\left(1+\frac{1}{m-1}\right)^{m-1} \longrightarrow 1+2em.$$

Also the step length is given explicitly by $1+1/(m-1)$.

In the randomized case the step length a_m^* is given implicitly as the value $a>1$ that minimizes the expression

$$\frac{a^m-1}{(a-1)\ln a}$$

[KRT93]; once having computed a_m^* the competitive ratio c_m^* is obtained by

$$c_m^* = 1+2\frac{(a_m^*)^m-1}{m(a_m^*-1)\ln a_m^*}.$$

It is not at all clear how c_m^* behaves as m goes to infinity. In the following we make use of the Lambert function W which is given by the functional equation $W(x)e^{W(x)}=x$.

Theorem 4.1 *The optimal step length of a randomized strategy to search on m rays is approximately $1+\alpha/m$, where $\alpha = 2+W(-2e^{-2}) \sim 1.59$ minimizes the function $(e^x-1)/x^2$ and the optimal competitive ratio converges to*

$$1 + 2\frac{e^\alpha - 1}{\alpha^2} m \sim 1 + 2 \cdot 1.544\, m$$

as m goes to infinity. Moreover, for every $m \geq 2$, $1 + 2 \cdot 1.545\, m$ is an upper bound on the competitive ratio c_m^*.

Proof: In order to prove the claim we write a_m^* as $a_m^* = 1 + \alpha_m/m$ and consider the function

$$f_m(x) = 1 + 2\frac{(1 + x/m)^m - 1}{m((1 + x/m) - 1)\ln(1 + x/m)} \quad = \quad 1 + 2m\frac{(1 + x/m)^m - 1}{xm\ln(1 + x/m)}.$$

Since $(1 + x/m)^m$ converges uniformly to e^x and $m\ln(1 + x/m)$ converges uniformly to x on the interval $[1, 2]$ as m goes to infinity, we obtain that f_m converges uniformly to

$$f_\infty(x) = 1 + 2m\frac{e^x - 1}{x^2}$$

on the interval $[1, 2]$. The unique minimum α of f_∞ is given by the equation

$$(2 - \alpha)e^\alpha - 2 = 0$$

and we obtain $\alpha = 2 + W(-2e^{-2}) \sim 1.59$. Since $\alpha \in (1, 2)$ and f_m converges uniformly to f_∞ on $[1, 2]$, there is an $m_0 \geq 1$ such f_m has a minimum in $(1, 2)$, for all $m \geq m_0$. Since f_m has exactly one minimum α_m, α_m converges to α as m goes to infinity.

In order to see that $1 + 2 \cdot 1.545\, m$ is an upper bound on the competitive ratio c_m^* we observe that the derivative of f_m w.r.t. m is positive. Hence, $f_{m+1}(x) \geq f_m(x)$, for $x \geq 1$ and, therefore, the minimum of $f_{m+1}(x)$ is at least as large as the minimum of $f_m(x)$. This implies that the limit of the minima is an upper bound. □

The above theorem implies that the time needed by the optimal randomized strategy is only $(e^\alpha - 1)/\alpha^2/e \sim 0.568$ times the time needed by the optimal deterministic strategy to find the goal on worst case inputs as m goes to infinity.

5 Conclusions

We present a simple proof that

$$c_m^* = 1 + \frac{2}{m} \min_{a>1} \frac{a^m - 1}{(a - 1)\ln a}$$

is a lower bound for randomized search strategies on m rays. In particular, we show in our proof that there is an optimal strategy that is periodic. We also give a lower bound on the convergence rate of the competitive ratio to c_m^* if an upper D on the distance to the goal is known. Finally, we compute the constant of proportionality for c_m^* and show that $c_m^* \sim 1 + 2 \cdot 1.544\, m$.

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