Parallel Searching on m Rays*

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Abstract

In this paper we investigate parallel searching on m concurrent rays. We assume that a target t is located somewhere on one of the rays and that a group of m point robots has to reach t. Furthermore, we assume that the robots have no way of communicating over distance. Given a strategy S we are interested in the competitive ratio which is defined as the ratio of the time needed by the robots using S and the time needed if the location of t is known in advance.

If a lower bound on the distance to the target is known, then there is a simple strategy which achieves a competitive ratio of 9 — independent of m. We show that even in the case $m = 2$ there is a lower bound of 9 on the competitive ratio for two large classes of strategies. Moreover, we show that a lower bound of 9 for $m = 2$ implies a lower bound of 9 for $m > 2$ — as is to be expected.

If the minimum distance to the target is not known in advance, then we show a lower bound on the competitive ratio of $1 + 2(k+1)^{k+1}/k^k$ where $k = \lceil \log m \rceil$. We also present a strategy that achieves this competitive ratio.

1 Introduction

Searching for a target is an important and well studied problem in robotics. In many realistic situations the robot does not possess complete knowledge about its environment, for instance, the robot may not have a map of its surroundings, or the location of the target may be unknown [BRS93, CL93, DHS95, DI94, IK95, Kle92, Kle94, LOS95, MI94, PY89].

The search of the robot can be viewed as an *on-line* problem since the robot's decisions about the search are based only on the part of its environment that it has seen so far. We use the framework of *competitive analysis* to measure the performance of an on-line search strategy S [ST85]. The *competitive ratio* of S is defined as the maximum of the ratio of

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the distance traveled by a robot using S to the optimal distance from its starting point to the target, over all possible locations in the environment of the target.

A problem with paradigmatic status in this framework is searching on m concurrent rays. Here, a point robot or—as in our case—a group of point robots is imagined to stand at the origin of m concurrent rays. One of the rays contains the target t whose distance to the origin is unknown. A robot can detect t only if it stands on top of it. It can be shown that an optimal strategy for one robot is to visit the rays in cyclic order, increasing the step length each time by a factor of $m/(m-1)$ if it starts with a step length of 1 [BYCR93, Gal80]. The competitive ratio C_m achieved by this strategy is given by

$$1 + 2\frac{m^m}{(m-1)^{m-1}}.$$

The lower bound for searching in m rays has proved to be a very useful tool for proving lower bounds for searching in a number of classes of simple polygons, such as star-shaped polygons [LO96], generalized streets [DI94, LOS96], HV-streets [DHS95], and θ-streets [DHS95, Hip94].

In this paper we are interested in obtaining upper and lower bounds for the competitive ratio of *parallel searching* on m concurrent rays. This problem has been adressed before in two contexts.

The first context is the on-line construction of hybrid algorithms the setting of which can be described as follows [KMSY94]: We are given a problem Q and m approaches to solving it. Each approach is implemented by an algorithm which is called a *basic* algorithm. We have a computer with $k \leq m$ disjoint memory areas which can be used to run one basic algorithm and to store the results of its computation. Only a single basic algorithm can be run by the computer at a given time. It is not known in advance which of the algorithms solves the problem Q—although we assume that there is at least one—or how much time it takes to compute a solution. In the worst case only one algorithm solves Q whereas the others do not even halt on Q. One way to solve Q is to construct a hybrid algorithm that uses the basic algorithms in the following way. A basic algorithm is run for some time, and then the computer switches to another algorithm and so on until Q is solved. If $k < m$, then there is not enough memory to save all the intermediate results. So sometimes the current intermediate results have to be discarded and to be recomputed later from the scratch.

A different way to look at this problem is to assume that we are given k robots that have to search on m rays for a target t with $k < m$. Each ray corresponds to a basic algorithm, and a robot corresponds to a memory area. At any time we are allowed to move only one robot. Discarding intermediate results of the basic algorithm A corresponds to moving the robot on the ray corresponding to A back to the origin.

Kao *et al.* [KMSY94, Yin94] present an algorithm for the above problem that achieves an optimal competitive ratio of

$$k + 2\frac{(m-k+1)^{m-k+1}}{(m-k)^{m-k}}$$

which is, of course, also the competitive ratio of searching with k robots on m rays if only one robot is allowed to move at a time.

In the second context a group of m point robots searches for the target. Again neither the ray containing the target nor the distance to the target are known. Now all the robots have to reach the target and the only way two robots can communicate is if they meet, that is, they have no communication device. We are going to use this model in our paper. Baeza-Yates and Schott investigate searching on the real line, that is, $m = 2$ [BS95]. They present two strategies both of which achieve a competitive ratio of 9. They also consider searching for a target line in the plane with multiple robots and present symmetric and asymmetric strategies. However, the question of optimality, that is, corresponding lower bounds, is not considered.

In this paper we investigate search strategies for parallel searching on m concurrent rays. If a lower bound on the distance to the target is known, then there is a simple strategy that achieves a competitive ratio of 9—independent of m. We show that even in the case $m = 2$ there is a matching lower bound of 9 on the competitive ratio of two large classes of strategies. Moreover, we show that, for all strategies, a lower bound of 9 for $m = 2$ implies a lower bound of 9 for $m > 2$—as is to be expected.

If the minimum distance to the target is not known in advance, then we show a lower bound on the competitive ratio of $1 + 2(k+1)^{k+1}/k^k$ where $k = \lceil \log m \rceil$. We also present a strategy that achieves this competitive ratio.

The paper is organized as follows. In the next section we present some definitions and preliminary results. In particular, we present three strategies to search on the line ($m = 2$), each with a competitive ratio of 9 and show that one of them can also be used to search on m rays with the same competitive ratio. In Section 3 we show a matching lower bound of 9 for two large classes of strategies. Finally, in Section 4 we present an optimal algorithm to search on m rays if there is no lower bound on the distance to the target.

2 Preliminaries

In the following we consider the problem of a group of m robots searching for a target of unkown location on m rays in parallel. The robots have the same maximal speed which we assume without loss of generality to be 1 distance unit per time unit. If the robots have unbounded speed, then the time to find the target (both off-line and on-line) can be made arbitrarily small. The speed of a robot may be positive (if it moves away from the origin) or negative (if it moves towards the origin).

Let S be a strategy for parallel searching on m rays and $T_S(D)$ the maximum time the group of robots needs to find and reach a target placed at a distance of D if it uses strategy S. Since the maximum speed of a robot is one, the time needed to reach the target if the position of the target is known is D time units. The competitive ratio is now defined as the maximum of $T_S(D)/D$, over all $D \geq 0$. In some applications a lower bound D_{min} on the distance to the target may be known. If such a lower bound exists, then we assume without loss of generality that $D_{min} = 1$. It will turn out that the existence of D_{min} leads to a drastically lower competitive ratio if $m > 2$.

We define different classes of possible strategies to search on m rays in parallel. We say a strategy is *monotone* if, at all times, all the robots (that do not know the location of the target) have non-negative speed. We say a strategy is *full speed* if all the robots travel at a speed of 1 or -1 at all times. We say a strategy is *symmetric* if, at all times, all the robots (that do not know the location of the target) have the same speed.

We illustrate the different types of strategies for $m = 2$. The optimal monotone strategy is for each robot to travel at a speed of $1/3$ on each ray. After one robot has found the target, it runs back to fetch the other. This leads to a competitive ratio of 9. This strategy is described in [BS95]. In the next section we show a lower bound of 9 on the competitive ratio of monotone strategies. The optimal (full-speed) symmetric strategy is for each robot to double the distance that has been explored before and then to return to the origin. This strategy can only be applied if a lower bound on the distance to the target is known. It achieves a competitive ratio of 9. Again this strategy is described in [BS95] and we show a lower bound of 9 on the competitive ratio of symmetric strategies in the next section. Finally, an asymmetric strategy is for both robots to walk together and to use the optimal strategy for one robot to search on two rays. This again yields a competitive ratio of 9.

3 With Lower Bound on the Minimum Distance

In this section we consider the problem of a group of m robots searching for a target of unkown location on m rays in parallel; we assume that a lower bound on the distance from the origin to the target of $D_{min} = 1$ is known.

Initially we study the parallel search problem on two rays and prove a lower bound on the competitive ratio for monotone strategies. We can view the two rays as being the positive and negative parts of the real line with the two robots initially placed at the origin. As time passes the robots move continuously and monotonically with some speed along the line until one of them finds the target. This robot now travels at full speed to the other robot and communicates to it the location of the target and they both return to this target point.

Let $v_1(T)$ and $v_2(T)$ be the *average* speeds of the two robots at time T, i.e., the distance of the robot to the origin at this time divided by the time. It is clear that a search strategy is completely specified by the two average speed functions. We have the following lower bound.

Lemma 3.1 *There is no monotone strategy that achieves a better competitive ratio than 9 to search on two rays in parallel.*

Proof: The proof uses an adversary to place the target point in order to maximize the competitive ratio.

First of all we realize that the two robots will not both go in the positive or negative direction since then the adversary can place the target on the ray not visited by the robots, yielding an infinite competitive ratio.

Hence, the two robots go in different directions. Consider the function $v_{min}(T)$ specified by

$$v_{min}(T) = \min_{t \in [0,T]} \{\min\{v_1(t), v_2(t)\}\}.$$

The function $v_{min}(T)$ is a monotone, non-increasing function with $v_{min}(T) \geq 0$, for all $T > 0$. It therefore has a limit. Let $v_{min} = \lim_{T \to \infty} v_{min}(T)$. From the definition of v_{min}, we know that $v_1(T) \geq v_{min}$ and $v_2(T) \geq v_{min}$, for all $T > 0$.

Suppose first that $v_{min} < 1/9$, then there is some point in time T where either $v_1(T) < 1/9$ or $v_2(T) < 1/9$. Assume without loss of generality that it occurs for v_1. Placing the target at the corresponding point on the ray yields a competitive ratio > 9, since it takes the robot at least 9 time units to find the target, proving our claim.

If $v_{min} \geq 1/9$, then, for any $\epsilon > 0$, there is a $T_D \geq 9$ such that either $v_1(T_D) \leq v_{min} + \epsilon$ or $v_2(T_D) \leq v_{min} + \epsilon$. Assume without loss of generality that for some specific ϵ it holds for $v_1(T_D)$. This means that $v_1(T_D)T_D \geq \frac{1}{9}9 \geq 1 = D_{min}$, so the adversary places the target at the point $D = v_1(T_D)T_D$. A lower bound on the competitive ratio C of the strategy can now be expressed as

$$C \geq \frac{T_D + 2T_R}{D},$$

where T_R denotes the time for the robot that found the target to reach the second robot at full speed. Since they both have to return to the target this adds an extra term T_R. Because of the monotonicity of the strategy we can express the time T_R by

$$T_R = D + v_2(T_D + T_R)(T_D + T_R).$$

This is because the robot that finds the target goes towards the other robot at full speed travelling first the distance D and then the distance $v_2(T_D + T_R)(T_D + T_R)$ until they meet. We have that

$$T_R = \frac{D + v_2(T_D + T_R)T_D}{1 - v_2(T_D + T_R)},$$

and hence that

$$
\begin{aligned}
C &\geq \frac{T_D + 2\frac{v_1(T_D)T_D + v_2(T_D+T_R)T_D}{1-v_2(T_D+T_R)}}{v_1(T_D)T_D} \\
&= \frac{1}{v_1(T_D)} + \frac{2}{1-v_2(T_D+T_R)} + \frac{2}{v_1(T_D)}\frac{v_2(T_D+T_R)}{(1-v_2(T_D+T_R))} \\
&\geq \frac{1}{v_{min}+\epsilon} + \frac{2}{1-v_{min}} + \frac{2}{(v_{min}+\epsilon)}\frac{v_{min}}{(1-v_{min})} \\
&= \frac{1-\epsilon/(v_{min}+\epsilon)}{v_{min}} + \frac{2}{1-v_{min}} + \frac{2-2\epsilon/(v_{min}+\epsilon)}{1-v_{min}} \\
&= \frac{1}{v_{min}} + \frac{4}{1-v_{min}} - \frac{\epsilon(v_{min}+1)}{v_{min}(v_{min}+\epsilon)(1-v_{min})} \\
&\geq 9 - \frac{18\epsilon}{1+3\epsilon},
\end{aligned}
$$

since the above expression is minimized for $v_{min} = \frac{1}{3}$ and it tends to 9 as ϵ tends to 0. □

Next we look at symmetric strategies. The following lemma shows that only symmetric full speed strategies need to be considered.

Lemma 3.2 *For all $\varepsilon > 0$ and all strategies S, there is a full speed strategy S' such that the competitive ratio of S' is at most $(1+\varepsilon)$ times the competitive ratio of S.*

Proof: Divide the time into small intervals of length δ. If strategy S has average speed v in an interval I, then let S' be the full-speed strategy that first travels back for $(1-v)\delta/2$ time units and then forward for $(1+v)\delta/2$ time units in I. At the end of I the robot is at the same position as if it had used S. By letting δ go to 0, the claim follows. □

The previous lemma tells us that we can simulate any strategy with a full speed strategy and therefore we only have to consider full speed strategies when we have a lower bound on the distance to the target.

In the following we show a lower bound for symmetric full speed strategies. We start with the simpler case when $m = 2$ and consider the general version later. In the proof of the lower bound for the simple case, we will use a technical result by Schuierer [Sch97] and therefore present it here.

Let $X = (x_0, x_1, \ldots)$ be a sequence of positive real numbers and let p and d be integers such that $0 \leq d < p$. We define

$$a_d = \overline{\lim_{n \geq 0}} (x_{np+d})^{\frac{1}{np+d}}$$

and

$$a = \overline{\lim_{n \geq 0}} (x_n)^{\frac{1}{n}}.$$

For $p, k \geq 1$, we denote the $(k+1)$-dimensional convex cone spanned by the set of vectors $\{(x_{ip}, x_{ip+1}, \ldots, x_{ip+k}) \mid 0 \leq i < \infty\}$, by $W_{p,k}(X)$, that is,

$$W_{p,k}(X) \quad = \quad \{Y \mid Y = \sum_{i=0}^{n} \beta_i (x_{ip}, x_{ip+1}, \ldots, x_{ip+k}), \text{ with } 0 \leq \beta_i \text{ and } 0 \leq n < \infty\}.$$

We denote the closure of $W_{p,k}(X)$ by $\overline{W}_{p,k}(X)$. Our results are based on the following crucial observation about the geometry of the convex cones $W_{p,k}(X)$ [Sch97].

Lemma 3.3 *Let $X = (x_0, x_1, \ldots)$ be a sequence of positive numbers and p an integer. If $a_d = a \in (0, \infty)$, for all $0 \leq d < p-1$, and $\overline{\lim}_{n \geq \infty} x_{n+1}/x_n < \infty$, then there exist p positive numbers $\gamma_0, \gamma_1, \ldots, \gamma_{p-1}$ that only depend on X such that, for all $k \geq 1$,*

$$\left(\gamma_0, \gamma_1 a, \gamma_2 a^2, \ldots, \gamma_{p-1} a^{p-1}, \gamma_0 a^p, \gamma_1 a^{p+1}, \ldots, \gamma_{k \bmod p} a^k\right) \in \overline{W}_{p,k}(X).$$

The above lemma provides the basis for our lower bound proofs. As a first step we show a lower bound on the competitive ratio of a symmetric strategy in terms of certain positive sequences.

Lemma 3.4 *Let $\mathcal{Z}$ be the set of infinite positive sequences $Z = (z_0, z_1, z_2, \ldots)$ with $z_{2k} > z_{2k-2}$ and $z_{2k+1} > z_{2k-1}$, for all $k \geq 1$. If S is a symmetric strategy, then the competitive ratio C_S of S is at least*

$$\inf_{Z \in \mathcal{Z}} \sup_{k \geq 1} \left\{ 1 + 2 \max \left\{ \frac{z_{2k} + z_{2k+1}}{z_{2k-1}}, \frac{z_{2k+2} + z_{2k+3}}{z_{2k+2} - z_{2k}} \right\} \right\}.$$

Proof: Since the strategy is symmetric, the two robots will use the same local strategy to search its own ray. Furthermore, by Lemma 3.2 we can assume that it is a full speed strategy. We can model a full speed strategy for a robot by saying that it first moves a distance x_0 forward along the ray at full speed, then it moves a distance y_0 backwards at full speed, then a distance x_1 forward, a distance y_1 backward, and so on. When one of the robots finds the target, it runs back at full speed until it meets the other robot, and they both run to the target at full speed.

The proof uses an adversary to place the target point in order to maximize the competitive ratio.

We say that a robot is in *step k* when it moves forward and backward the $k + 1^{\text{st}}$ time. Let L_k denote the distance to the origin of the turning point where the robot begins step k and let U_k denote the distance to the origin of the turning point where the robot starts to move backwards during step k (see Figure 1). We have that

$$L_0 \quad = \quad 0,$$

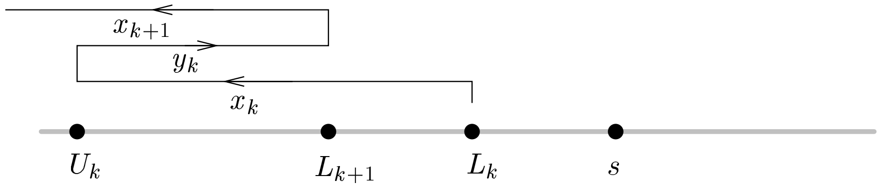

Figure 1: The definition of U_k and L_k.

$$
\begin{aligned}
L_k &= L_{k-1} + x_{k-1} - y_{k-1} &&= \sum_{i=0}^{k-1} x_i - y_i, \\
U_k &= L_k + x_k &= x_k + \sum_{i=0}^{k-1} x_i - y_i &= y_k + \sum_{i=0}^{k} x_i - y_i.
\end{aligned}
$$

The total time that the robot has travelled when it completes step k is

$$
T_k = \sum_{i=0}^{k} x_i + y_i.
$$

First of all, we can assume that $y_k \leq U_k$ for all $k \geq 0$, that is, a robot always stays on the same ray, since if this does not hold, we can exchange the strategy for an equivalent one where, if the two robots meet at the origin, they exchange places and continue on their own ray instead of the other robot's ray.

Secondly, we can assume that $U_{k-1} < U_k$, since otherwise, the strategy will not explore any new part of the ray during step k, and we can exchange the strategy for another equivalent one, where the assumption holds. In particular, we can assume that $y_k < x_{k+1}$, for all $k \geq 0$.

Assume that the target is placed at distance D with $1 = D_{min} \leq D$ and $U_{k-1} < D \leq U_k$, for some $k \geq 0$. Without loss of generality, we can assume that $U_{-1} = 1$, if $U_0 > 1$, since we can set $x_{-1} = 1$, $y_{-1} = \epsilon$, and $x_0 = U_0 - 1 + \epsilon$. This means that one of the robots will find the target during step k. If T_R again denotes the time for the robot that found the target to reach the second robot at full speed, then the competitive ratio for this placement is given by

$$
\begin{aligned}
C_D &= \frac{T_{k-1} + (D - L_k) + 2T_R}{D} = 1 + \frac{T_{k-1} - L_k + 2T_R}{D} \\
&= 1 + 2\frac{\sum_{i=0}^{k-1} y_i + T_R}{D},
\end{aligned}
$$

since $D - L_k$ is the time the robot needs to reach the target in step k. As both of them have to return to the target, an extra factor of two for T_R is needed.

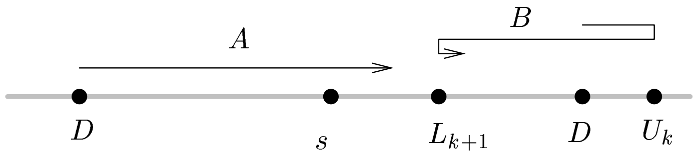

Figure 2: Robot A misses robot B if $D + L_{k+1} > U_k - D + U_k - L_{k+1}$.

We will only consider two possible placements for the target and let the adversary choose the one that maximizes the competitive ratio. For the first placement of the target, we assume that the other robot is reached while it is still in step k. (This is the best case for the strategy.) Since we place the target in the interval $]U_{k-1}, U_k]$, the ratio C_k^1 of the time needed by the strategy to the optimal time is given by

$$\begin{aligned} C_k^1 &= \sup_{D \in]U_{k-1}, U_k]} \left\{ 1 + 2\frac{\sum_{i=0}^{k-1} y_i + T_R}{D} \right\} = 1 + 2\frac{\sum_{i=0}^{k-1} y_i + T_R}{U_{k-1}} \\ &= 1 + 2\frac{\sum_{i=0}^{k-1} y_i + U_k}{U_{k-1}} \end{aligned}$$

as it can be easily seen that $T_R = \frac{D+(U_k-D)+U_k}{2} = U_k$ in this case. This implies that the adversary places the target right after the point U_{k-1}.

In the second possible placement the robot that finds the target just fails to reach the other robot during step k. Since both robots travel at full speed, the earliest they can meet is during the step $k+1$. In order for robot A to miss robot B the distance $D + L_{k+1}$ from the point where A finds the target to the forward turning point of B at the beginning of step $k+1$ has to be larger than the distance from the point D on B's ray to B's backward turning point at U_k plus the distance from U_k to L_{k+1} (see Figure 2), that is,

$$L_{k+1} + D > U_k - D + U_k - L_{k+1} = U_k - D + y_k,$$

i.e., when $D > y_k$. The placement of the target is therefore restricted to the interval $]y_k, U_k]$, which implies that the adversary places the target right after y_k. The ratio C_k^2 of the time needed by the strategy to the optima is given by

$$\begin{aligned} C_k^2 &= \sup_{D \in]y_k, U_k]} \left\{ 1 + 2\frac{\sum_{i=0}^{k-1} y_i + T_R}{D} \right\} = 1 + 2\frac{\sum_{i=0}^{k-1} y_i + T_R}{y_k} \\ &= 1 + 2\frac{\sum_{i=0}^{k-1} y_i + x_{k+1} + U_k + y_k - y_k}{y_k} = 1 + 2\frac{\sum_{i=0}^{k} y_i + U_{k+1}}{y_k} \end{aligned}$$

since $T_R = \frac{D+(U_k-D)+y_k+x_{k+1}+U_{k+1}}{2} = \frac{y_k+x_{k+1}+U_{k+1}+U_k}{2} = x_{k+1} + U_k$ again by a simple geometric argument.

It may happen that independently of where the target is placed in the interval $]U_{k-1}, U_k]$ the robot that finds the target will never miss the other robot during step k. This means that $y_k \geq U_k$ because placing the target on the point U_k and requiring that the other robot is met during step k implies that the two robots meet at the origin. On the other hand, we know from before that $y_k \leq U_k$, and hence, that $y_k = U_k$.

Now, we consider the situation if the adversary places the target right after the point U_k. We obtain a competitive ratio of

$$1 + 2\frac{\sum_{i=0}^{k} y_i + T_R}{U_k} \quad = \quad 1 + 2\frac{\sum_{i=0}^{k} y_i + U_{k+1}}{y_k} = C_k^2.$$

Hence, C_k^2 is a lower bound for the competitive ratio independent of the fact whether the robot that finds the target can miss or cannot miss the other robot in step k. The adversary can now choose between C_k^1 and C_k^2 to maximize the competitive ratio. Hence, the best competitive ratio for any symmetric strategy is bounded below by

$$C \quad = \quad \sup_{k \geq 0} \max\{C_k^1, C_k^2\}.$$

If we make the variable substitutions $z_{2k} = \sum_{i=0}^{k} y_i$ and $z_{2k+1} = U_{k+1}$, for all $k \geq 0$, and let $Z = (z_0, z_1, \ldots)$, we can express $\max\{C_k^1, C_k^2\}$, $k \geq 2$ as the functional

$$F_{k-1}(Z) \quad = \quad 1 + 2\max\left\{\frac{z_{2k-2} + z_{2k-1}}{z_{2k-3}}, \frac{z_{2k} + z_{2k+1}}{z_{2k} - z_{2k-2}}\right\}$$

where $z_{2i} > z_{2i-2}$ and $z_{2i+1} > z_{2i-1}$, for all $i \geq 1$. This proves the claim. □

Now we show the main result on symmetric strategies.

Theorem 3.5 *There is no symmetric strategy that achieves a better competitive ratio than 9 to search on two rays in parallel.*

Proof: Let S be a symmetric strategy to search on two rays, let $\mathcal{Z}$ be the set of infinite positive sequences $Z = (z_0, z_1, z_2, \ldots)$ with $z_{2k} > z_{2k-2}$ and $z_{2k+1} > z_{2k-1}$, for all $k \geq 1$, and let

$$F_k(Z) = 1 + 2\max\left\{\frac{z_{2k} + z_{2k+1}}{z_{2k-1}}, \frac{z_{2k+2} + z_{2k+3}}{z_{2k+2} - z_{2k}}\right\},$$

for $Z \in \mathcal{Z}$. By Lemma 3.4 the competitive ratio C_S of S satisfies

$$C_S \quad \geq \quad \inf_{Z \in \mathcal{Z}} \sup_{k \geq 1} F_k(Z). \tag{1}$$

In the following we show that the above expression is bounded from below by 9. Let $Z = (z_0, z_1, z_2, \ldots)$ be a sequence in $\mathcal{Z}$. First assume that $a = \overline{\lim}_{n\to\infty}(z_n)^{1/n} > 1$. In this case we can show that

$$F_k(\gamma_0, \gamma_1 a, \gamma_0 a^2, \gamma_1 a^3, \ldots) \quad \leq \quad \sup_{0 \leq i < \infty} F_k(Z^{2i}) \tag{2}$$

where Z^{2i} is the suffix of Z that starts with element z_{2i}, that is, $Z^{2i} = (z_{2i}, z_{2i+1}, \ldots)$. We postpone the proof of Inequality 2 for the moment. The claim can now be seen as follows. Inequality 2 implies that

$$\sup_{0 \le k < \infty} \sup_{0 \le i < \infty} F_k(Z^{2i}) \ge \sup_{0 \le k < \infty} F_k(\gamma_0, \gamma_1 a, \gamma_0 a^2, \ldots).$$

Now, for all $k, i \ge 0$, $F_{k+i}(Z) = F_k(Z^{2i})$ and, therefore,

$$\begin{aligned}
\sup_{0 \le k < \infty} F_k(Z) &= \sup_{0 \le k, i < \infty} F_{k+i}(Z) = \sup_{0 \le k < \infty} \sup_{0 \le i < \infty} F_k(Z^{2i}) \\
&\ge \sup_{0 \le k < \infty} F_k(\gamma_0, \gamma_1 a, \gamma_0 a^2, \ldots).
\end{aligned}$$

It remains to compute a lower bound on $F_k(\gamma_0, \gamma_1 a, \gamma_0 a^2, \ldots)$ for $a > 1$ and $\gamma_0, \gamma_1 > 0$. Let $\gamma = \gamma_1/\gamma_0$; then,

$$\begin{aligned}
F_k(\gamma_0, \gamma_1 a, \gamma_0 a^2, \ldots) &= 1 + 2 \max \left\{ \frac{a^{2k} + \gamma a^{2k+1}}{\gamma a^{2k-1}}, \frac{a^{2k+2} + \gamma a^{2k+3}}{a^{2k+2} - a^{2k}} \right\} \\
&= 1 + 2 \max \left\{ \frac{a}{\gamma} + a^2, \frac{a^2 + \gamma a^3}{a^2 - 1} \right\}
\end{aligned}$$

which is minimized for $\gamma = a - 1/a$ and $a = \sqrt{2}$ yielding a value of 9, that is,

$$\sup_{0 \le k < \infty} F_k(Z) \ge 9,$$

for all $Z \in \mathcal{Z}$. The claim now follows by Inequality 1.

It remains to show Inequality 2. Note that if

$$\overline{\lim_{n \ge 0}} (z_{2n+1})^{1/(2n+1)} = a_1 \quad \ne \quad a_2 = \overline{\lim_{n \ge 0}} (z_{2n})^{1/(2n)},$$

then it is easy to see that $\overline{\lim}_{n \ge 0} z_{n+1}/z_n = \infty$. But if $\overline{\lim}_{n \ge 0} z_{n+1}/z_n = \infty$, then either $\overline{\lim}_{n \ge 0} z_{2n+2}/z_{2n+1} = \infty$ or $\overline{\lim}_{n \ge 0} z_{2n+1}/z_{2n} = \infty$. In the first case the first term of the maximum of F_k goes to infinity and in the second case the second. Therefore, we can assume that $a_1 = a_2 = a$ and $\overline{\lim}_{n \ge 0} z_{n+1}/z_n < \infty$, that is, Z satisfies all the conditions of Lemma 3.3, for $p = 2$. Let $k \ge 0$ be fixed. Lemma 3.3 now implies that there exist positive numbers γ_0 and γ_1 with

$$\left(\gamma_0, \gamma_1 a, \gamma_0 a^2, \ldots, \gamma_1 a^{2k+3} \right) \in \overline{W}_{2,2k+3}(Z).$$

This in turn implies, that, for any $\delta > 0$, there exists an integer $n \ge 0$ and non-negative numbers β_i, for $0 \le i \le n$, and such that the elements $z_{j,\delta}$ of the sequence Z_δ which is defined by

$$Z_\delta = \sum_{i=0}^{n} \beta_i Z^{2i}$$

satisfy

$$|z_{j,\delta} - \gamma_d a^j| < \delta$$

for $0 \le j \le 2k+3$, where $d = j \bmod 2$. Moreover, it is easy to see that if $X, Y \in \mathcal{Z}$ and $\alpha, \beta \ge 0$, then

$$F_k(\alpha X + \beta Y) \quad \le \quad \max\{F_k(X), F_k(Y)\} \tag{3}$$

since $(a+c)/(b+d) \le \max\{a/b, c/d\}$ if $a, b, c, d > 0$. We also have $F_k(\alpha X) = F_k(X)$, for $\alpha \ge 0$. Finally, note that $Z^{2i} \in \mathcal{Z}$ and, therefore,

$$Z_\delta = \sum_{i=0}^{n} \beta_i Z^{2i} \in \mathcal{Z}.$$

Hence,

$$\begin{aligned} F_k(Z_\delta) &= F_k(z_{0,\delta}, z_{1,\delta}, \ldots, z_{2k+3,\delta}, z_{2k+4,\delta}, \ldots) \\ &= F_k(\sum_{i=0}^{n} \beta_i Z^{2i}) \;\le\; \sup_{0 \le i < \infty} F_k(\beta_i Z^{2i}) \;=\; \sup_{0 \le i < \infty} F_k(Z^{2i}). \end{aligned}$$

Here, the inequality follows by an inductive application of Inequality 3; the last equality follows from the fact that $\beta_i \ge 0$. Therefore,

$$\sup_{0 \le i < \infty} F_k(Z^{2i}) \quad \ge \quad \lim_{\delta \to 0} F_k(Z_\delta) = F_k(\gamma_0, \gamma_1 a, \gamma_0 a^2, \ldots)$$

where the above equality follows from the fact that F_k is continuous in a neighbourhood of $(\gamma_0, \gamma_1 a, \gamma_0 a^2, \ldots)$ if $a > 1$.

Finally, we consider the case $a = 1$. In this case the inequality

$$\sup_{0 \le i < \infty} F_k(Z^{2i}) \ge F_k(Z_\delta)$$

still holds for all $\delta > 0$ and it is easy to see that since Z_δ is arbitrarily close to the sequence $(\gamma_0, \gamma_1, \gamma_0, \gamma_1, \ldots)$ the value $F_k(Z_\delta)$ tends to infinity as δ goes to 0 and strategies for which $a = 1$ have an unbounded competitive ratio. $\Box$

Theorem 3.6 *There is no monotone or symmetric strategy that achieves a better competitive ratio than 9 to search on m rays in parallel.*

Proof: We consider monotone strategies first. It is obvious that if the robots have further information about the location of the target, then the competitive ratio for a strategy that exploits this information does not increase. Assume that the robots know that the target is on one of the two first rays. They can all explore these rays monotonically in common.

Consider now the strategy we get by taking the furthest robot from the origin on each of the two rays. This strategy is a monotone strategy for two robots on two rays and by Lemma 3.1 no such strategy can do better than a competitive ratio of 9.

For symmetric strategies we can argue in a similar manner. Once a robot has found the target, the competitive ratio is bounded below by the time it takes to fetch all the other robots and to go back to the target. This ratio is bounded below by the time it takes for the robot that found the target to fetch *one* other robot and for it to go to the target. By Theorem 3.5 this competitive ratio is bounded below by 9. □

Now, there is a strategy that achieves a competitive ratio of 9 to search on m rays in parallel. The strategy is known as the *doubling strategy* and goes as follows. Each robot starts by going one unit at full speed on its ray and then goes back to the origin. Then they each go two units, four units, and so on, on their corresponding ray, always doubling the distance travelled and repeatedly going back to the origin. Once a robot finds the target, it goes back at full speed to the origin and waits there until the other robots reach it. It then communicates the location of the target to the other robots and they all move at full speed to the location. The competitive strategy of the doubling strategy, if the target is at distance D from the origin is

$$\begin{aligned} C &\leq \sup_{D \in]2^{k-1}, 2^k]} \left\{ \frac{2\sum_{i=0}^{k} 2^i + D}{D} \right\} \leq 1 + \sup_{D \in]2^{k-1}, 2^k]} \left\{ \frac{2\sum_{i=0}^{k} 2^i}{D} \right\} \\ &\leq 1 + \frac{2\sum_{i=0}^{k} 2^i}{2^{k-1}} = 1 + 2\frac{2^{k+1} - 1}{2^{k-1}} = 9 - \frac{1}{2^{k-2}} \leq 9. \end{aligned}$$

We have proved the following theorem.

Theorem 3.7 *The doubling strategy achieves a competitive ratio of 9 to search on m rays in parallel given a lower bound on the distance to the target.*

4 Without Lower Bound on the Minimum Distance

In this section we consider the problem of a group of m robots searching for a target of unkown location on m rays in parallel and no lower bound on the distance from the origin to the target is known.

We begin by presenting a strategy that achieves the competitive ratio

$$1 + 2\frac{(k+1)^{k+1}}{k^k},$$

where $k = \lceil \log m \rceil$. We then show that, in fact, no strategy can do better than this.

4.1 The Strategy

The optimal strategy is a monotone strategy where all the robots move, one on each ray, with a constant speed v. When one robot finds the target it searches for a robot at full speed to tell it where the target is located. Then they both go at full speed to search for two more robots and tell them the location of the target, and so on. After each step the number of robots that know the location of the target is doubled. Once all robots know the location, they all move to the target. Suppose the target is on some ray and at distance D from the origin. The strategy consists of *steps.* Step i starts when 2^i robots know the location to the target and ends when 2^{i+1} robots know the location to the target; that is, in step i the 2^i robots that currently know the position of the target chase 2^i of those robots that do not. Let T_i denote the time it takes to complete step i. It takes any of the robots D/v time to find the target, and when all robots know the location of the target, it takes them time T_F to go to the target. Hence, the competitive ratio of the strategy is

$$C = \frac{D/v + \sum_{i=0}^{k-1} T_i + T_F}{D},$$

where $k = \lceil \log m \rceil$.

The time that has passed in order to inform 2^i robots is $D/v + \sum_{j=0}^{i-1} T_j$. Hence, at the end of step $i-1$ all the robots have a distance of $D + v\sum_{j=0}^{i-1} T_j$ to the origin. The 2^i informed robots now start chasing 2^i uninformed robots. In order to reach them, they have to travel a distance of $D + v\sum_{j=0}^{i-1} T_j$ to reach the origin, a distance of $D + v\sum_{j=0}^{i-1} T_j$ to reach the location of the chased robot at the beginning of step i, and a distance of vT_i until the uninformed robots are finally reached. Hence, T_i is given by the equation

$$T_i = 2(D + v\sum_{j=0}^{i-1} T_j) + vT_i$$

and

$$T_i = \frac{2D + 2v\sum_{j=0}^{i-1} T_j}{1-v} = \frac{2D + 2v\sum_{j=0}^{i-2} T_j}{1-v} + \frac{2v}{1-v}T_{i-1} = T_{i-1}\frac{1+v}{1-v}.$$

This is a recurrence relation with the solution

$$T_i = T_0\left(\frac{1+v}{1-v}\right)^i = \frac{2D}{1-v}\left(\frac{1+v}{1-v}\right)^i.$$

We obtain

$$\begin{aligned}\sum_{i=0}^{k-1} T_i &= \frac{2D}{1-v}\sum_{i=0}^{k-1}\left(\frac{1+v}{1-v}\right)^i = \frac{2D}{1-v}\frac{\left(\frac{1+v}{1-v}\right)^k - 1}{\frac{1+v}{1-v} - 1} \\ &= \frac{D}{v}\left(\left(\frac{1+v}{1-v}\right)^k - 1\right).\end{aligned}$$

By the above considerations the time T_F it takes for the robots to get to the target is the time of the last robot that is informed to go to the origin $D + v\sum_{j=0}^{i-1} T_j$ plus the distance D to the target

$$T_F \;=\; D + v\sum_{i=0}^{k-1} T_i + D \;=\; D\left(\frac{1+v}{1-v}\right)^k + D.$$

So, the competitive ratio of the strategy is

$$\begin{aligned} C &= \frac{1}{v} + \frac{1}{v}\left(\frac{1+v}{1-v}\right)^k - \frac{1}{v} + \left(\frac{1+v}{1-v}\right)^k + 1. \\ &= 1 + \left(\frac{1}{v} + 1\right)\left(\frac{1+v}{1-v}\right)^k \\ &= 1 + \left(\frac{1}{\frac{1}{2k+1}} + 1\right)\left(\frac{1+\frac{1}{2k+1}}{1-\frac{1}{2k+1}}\right)^k \\ &= 1 + (2k+2)\left(\frac{2k+2}{2k}\right)^k \\ &= 1 + 2\frac{(k+1)^{k+1}}{k^k}, \end{aligned}$$

if we set the speed $v = \frac{1}{2k+1}$.

We have proved the following result.

Theorem 4.1 *There is a monotone strategy that achieves a competitive ratio of*

$$1 + 2\frac{(k+1)^{k+1}}{k^k},$$

where $k = \lceil \log m \rceil$, to search on m rays in parallel, if no lower bound on the distance to the target is known.

4.2 Lower Bound

We consider first the parallel search problem for monotone strategies and prove a lower bound on the competitive ratio for these strategies. As time passes the robots move continuously and monotonically with some speed along the rays until one of them has found the target. This robot now travels at full speed to one of the other robots and communicates to him the location of the target, they both travel to other robots to communicate the location of the target, and so on. When all robots know where the target is they all go to this target point.

Let $v_1(T), v_2(T), \ldots, v_m(T)$ be the average speeds of the m robots at time T, i.e., the coordinate position of the robot on its ray at this time divided by the time. It is clear that a search strategy is completely specified by the m average speed functions. We have the following lower bound.

Lemma 4.2 *There is no monotone strategy that achieves a better competitive ratio than*

$$1 + 2\frac{(k+1)^{k+1}}{k^k},$$

where $k = \lceil \log m \rceil$, *to search on* m *rays in parallel.*

Proof: As always, the proof uses an adversary to place the target point in order to maximize the competitive ratio. We will use a similar proof technique as in the proof of Lemma 3.1, except that instead of looking at what happens when the target is close to infinity we will look at the situation when the target is close to zero.

First of all we realize that the m robots will each go on different rays, since otherwise, the adversary can place the target on a ray not visited by the robots, yielding an infinite competitive ratio.

Let $v_{max} = \lim_{T \to 0} \max\{v_i(T)\}$ and $v_{min} = \lim_{T \to 0} \min\{v_i(T)\}$, for $1 \le i \le m$. The limits exist since if they did not, then the momentary speed for some robot in some time interval close to zero would exceed 1. Without loss of generality we can assume that $\lim_{T \to 0} v_1(T) = v_{min}$ and $\lim_{T \to 0} v_2(T) = v_{max}$, otherwise we reenumerate the avarage speed functions so that the above holds. We immediately have that $v_{min} \le v_{max}$.

The proof divides into two cases, one slightly awkward special case and the general case. We begin by handling the special case.

Assume in the first case that $v_{max} = 1$. If this is the case, then for every $\epsilon > 0$, there is a $T_\epsilon > 0$ such that $v_2(T) \ge 1 - \epsilon$, for all $T \le T_\epsilon$. The adversary chooses a point in time T_D such that $T_D < \frac{T_\epsilon}{1+2\frac{(k+1)^{k+1}}{k^k}}$, where $k = \lceil \log m \rceil$ and places the target at distance $v_1(T_D)T_D$ on the first ray. Let robot r be the robot that chooses to chase robot 2 and T_2 the time needed for robot r to catch robot 2. Clearly, the competitive ratio for any monotone strategy is bounded by the time T_2. Assume first that robot 2 is caught before time T_ϵ, then

$$C \ge \frac{T_2}{v_1(T_D)T_D} \ge \frac{\frac{v_1(T_D)T_D + v_2(T_D+T^*+T_2)T_D}{1 - v_2(T_D+T^*+T_2)}}{v_1(T_D)T_D} \ge 1 + \frac{1-\epsilon}{\epsilon v_1(T_D)} \ge 1 + \frac{1-\epsilon}{\epsilon},$$

which can be made arbitrarily large. The above holds since

$$T_2 \ge v_1(T_D)T_D + v_2(T_D + T^* + T_2)(T_D + T_2),$$

where T^* is the time that passes between when the target is found and when robot r starts chasing robot 2. If robot 2 is caught after time T_ϵ, then

$$C \ge \frac{T_D + T^* + T_2}{v_1(T_D)T_D} \ge \frac{T_\epsilon}{v_1(T_D)T_D} > 1 + 2\frac{(k+1)^{k+1}}{k^k},$$

since $v_1(T_D) \le 1$. This proves the competitive ratio for the special case.

Assume for the second case that $v_{max} < 1$, then, for every $\epsilon > 0$, there is a $T_\epsilon > 0$, such that $v_{min} - \epsilon \le v_1(T) \le v_{min} + \epsilon < 1$ and $v_{min} - \epsilon \le v_i(T) \le 1$, for all $T \le T_\epsilon$ and $2 \le i \le m$. The adversary places the target at a point $D = v_1(T_D)T_D$ on ray 1, T_D being chosen so that $T_D < \frac{T_\epsilon}{2+4\frac{(k+1)^{k+1}}{k^k}}$, for $k = \lceil \log m \rceil$. Once the first robot has found the target, we can designate the robots by two types. Robots of the first type are called the *hunters*, i.e., the robots that currently know the position of the target and are chasing after other robots in order to convey this information. The remaining robots are called the *prey*, and these continue on their respective ray using their monotone strategy. Initially at time T_D, only the first robot is a hunter, all the others are prey.

Denote by T_n, the first time when exactly n of the m robots are hunters. If $T_m > T_\epsilon/2$, then the competitive ratio exceeds

$$\frac{T_m}{v_1(T_D)T_D} > \frac{T_\epsilon}{2v_1(T_D)T_D} > 1 + 2\frac{(k+1)^{k+1}}{k^k},$$

for $k = \lceil \log m \rceil$, proving our claim.

If $T_m \le T_\epsilon/2$, then we use induction to prove the following

$$T_n \quad \ge \quad T_D \left(1 + 2\frac{v_{min} - \epsilon}{1 - v_{min} + \epsilon}\right)^{\lceil \log n \rceil}.$$

The base case $n = 1$ follows directly since we know that $T_1 = T_D$.

For $n > 1$, we prove the claim as follows. The time T_n is of course non-decreasing in n. Consider now the point in time T_n, for some specific n. At some time prior to T_n, at least $\lceil n/2 \rceil$ robots are hunters. Assume otherwise, i.e., that there are $n' < \lceil n/2 \rceil$ hunters at any time prior to T_n. Then the largest number of robots that can become hunters at time T_n is $2n' < n$, since each hunter can produce at most one new hunter at time T_n. This is a contradiction since at time T_n we have n hunters. For the same reason, there is a hunter and a prey that meet at a time T no earlier than $T_{\lceil n/2 \rceil}$ and then one of them, say robot i, chases some other prey, say robot j, and catches it at a time T' no later than T_n. Let $T_R = T' - T$. We have that

$$T_n \quad \ge \quad T_{\lceil n/2 \rceil} + T_R,$$

where

$$\begin{aligned} T_R \quad &\ge \quad \min_{T_{\lceil n/2 \rceil} \le T \le T_n} \{v_i(T)T + v_j(T+T_R)(T+T_R)\} \\ &\ge \quad \min_{T \le T_n} \{v_i(T)\} T_{\lceil n/2 \rceil} + \min_{T \le T_n} \{v_j(T+T_R)\} (T_{\lceil n/2 \rceil} + T_R) \end{aligned}$$

Hence,

$$\begin{aligned} T_R \quad &\ge \quad T_{\lceil n/2 \rceil} \frac{\min_{T \le T_n} \{v_i(T)\} + \min_{T \le T_n} \{v_j(T+T_R)\}}{1 - \min_{T \le T_n} \{v_j(T+T_R)\}} \\ &\ge \quad 2T_{\lceil n/2 \rceil} \frac{v_{min} - \epsilon}{1 - v_{min} + \epsilon}, \end{aligned}$$

since $T_n + T_R \leq 2T_m \leq T_\epsilon$.

From above we have that

$$\begin{aligned} T_n &\geq T_{\lceil n/2 \rceil} + 2T_{\lceil n/2 \rceil} \frac{v_{min} - \epsilon}{1 - v_{min} + \epsilon} &\geq T_{\lceil n/2 \rceil} \left(1 + 2\frac{v_{min} - \epsilon}{1 - v_{min} + \epsilon}\right) \\ &\geq T_1 \left(1 + 2\frac{v_{min} - \epsilon}{1 - v_{min} + \epsilon}\right)^{\lceil \log n \rceil} &= T_D \left(1 + 2\frac{v_{min} - \epsilon}{1 - v_{min} + \epsilon}\right)^{\lceil \log n \rceil}, \end{aligned}$$

by our induction hypothesis.

So, after T_m time, all robots are hunters. Assume that the last robot to become a hunter was robot l. If more than one robot became hunters at this time, we assume that robot l is the one with minimum average speed at time T_m. The competitive ratio of any monotone strategy can now be expressed by

$$\begin{aligned} C &\geq \frac{T_m + v_l(T_m)T_m + D}{D} = 1 + \frac{T_m + v_l(T_m)T_m}{v_1(T_D)T_D} \\ &\geq 1 + \left(1 + 2\frac{v_{min} - \epsilon}{1 - v_{min} + \epsilon}\right)^{\lceil \log m \rceil} \frac{1 + v_{min} - \epsilon}{v_{min} + \epsilon} \\ (k = \lceil \log m \rceil) \quad &= 1 + \left(\frac{1 + v_{min} - \epsilon}{1 - v_{min} + \epsilon}\right)^k \frac{1 + v_{min} - \epsilon}{v_{min} + \epsilon} \\ &= 1 + \frac{(1 + v_{min})^{k+1}}{v_{min}(1 - v_{min})^k} - \Theta(\epsilon) \\ &\geq 1 + \frac{(1 + \frac{1}{2k+1})^{k+1}}{\frac{1}{2k+1}(1 - \frac{1}{2k+1})^k} - \Theta(\epsilon) \\ &= 1 + 2\frac{(k+1)^{k+1}}{k^k} - \Theta(\epsilon), \end{aligned}$$

since the expression is minimized for $v_{min} = \frac{1}{2k+1}$ and it tends to $1 + 2\frac{(k+1)^{k+1}}{k^k}$ as ϵ tends to 0, which concludes the proof. $\square$

We are now in a position to prove the following theorem.

Theorem 4.3 *There is no strategy whatsoever that achieves a better competitive ratio than*

$$1 + 2\frac{(k+1)^{k+1}}{k^k},$$

where $k = \lceil \log m \rceil$, to search on m rays in parallel.

Proof: Any strategy can be specified by the average speed functions $v_1(T), \ldots, v_m(T)$ of the m robots. (In conjunction with information about whether a robot switches ray, at some point in time.) Given these average speed functions, an adversary can extract information about the time length that a robot moves monotonically along a ray. (This includes also the time that a robot stands still at the origin.) Let T_i, for $1 \le i \le m$ denote the time that robot i moves monotonically along a ray. Each T_i is greater than 0 since either the robot stands still or moves along some ray. If $T_\epsilon = \min_{1 \le i \le m}\{T_i\}$, then we let the adversary place the target on some ray, say ray 1, at distance $D = v_1(T_D)T_D$, such that $T_D < \frac{T_\epsilon}{1+2\frac{(k+1)^{k+1}}{k^k}}$, for $k = \lceil \log m \rceil$. If the strategy uses more than T_ϵ time, then the competitive ratio is trivially bounded from below by

$$1 + 2\frac{(k+1)^{k+1}}{k^k},$$

for $k = \lceil \log m \rceil$. If, on the other hand, the strategy uses less than T_ϵ time, then the strategy is monotone in the interesting time interval and we can apply Lemma 4.2, proving our claim. □

5 Conclusions

We considered search strategies for parallel search on m concurrent rays. We show that a straight forward generalization of the so called doubling strategy, from searching on the line to searching on m concurrent rays, yields a competitive ratio of 9 if a minimum distance from the origin to the target is known in advance. Furthermore, we prove that 9 is a lower bound on the competitive ratio for both monotone and symmetric strategies in this case.

We also prove a lower bound of

$$1 + 2\frac{(k+1)^{k+1}}{k^k}$$

on the competitive ratio, if a minimum distance from the origin to the target is not known in advance. Finally, we give a search strategy that achieves this ratio regardless of whether such a minimum distance is known or not, giving us an optimal search strategy in the latter case.

References

[Bec64] A. Beck. On the linear search problem. *Israel Journal of Math.*, 2:221–228, 1964.

[Bec65] A. Beck. More on the linear search problem. *Israel Journal of Math.*, 3:61–70, 1965.

[Bel63] R. Bellman. Problem 63-9*, an optimal search. *SIAM Review*, 5(2):274, 1963.

[BN70] A. Beck and D. Newman. Yet more on the linear search problem. *Israel Journal of Math.*, 8:419–429, 1970.

[BRS93] Margrit Betke, Ronald L. Rivest, and Mona Singh. Piecemeal learning of an unknown environment. In *Sixth ACM Conference on Computational Learning Theory (COLT 93)*, pages 277–286, July 1993.

[BS95] R. Baeza-Yates and R. Schott. Parallel searching in the plane. *Comput. Geom. Theory Appl.*, 5:143–154, 1995.

[BW72] A. Beck and P. Warren. The return of the linear search problem. *Israel Journal of Math.*, 10:169–183, 1972.

[BYCR93] R. Baeza-Yates, J. Culberson, and G. Rawlins. Searching in the plane. *Information and Computation*, 106:234–252, 1993.

[CL93] K-F. Chan and T. W. Lam. An on-line algorithm for navigating in an unknown environment. *International Journal of Computational Geometry & Applications*, 3:227–244, 1993.

[DHS95] A. Datta, Ch. Hipke, and S. Schuierer. Competitive searching in polygons—beyond generalized streets. In *Proc. Sixth Annual International Symposium on Algorithms and Computation*, pages 32–41. LNCS 1004, 1995.

[DI94] A. Datta and Ch. Icking. Competitive searching in a generalized street. In *Proc. 10th Annu. ACM Sympos. Comput. Geom.*, pages 175–182, 1994.

[Gal72] S. Gal. A general search game. *Israel Journal of Math.*, 12:32–45, 1972.

[Gal74] S. Gal. Minimax solutions for linear search problems. *SIAM Journal of Applied Math.*, 27(1):17–30, 1974.

[Gal80] S. Gal. *Search Games.* Academic Press, 1980.

[GC76] S. Gal and D. Chazan. On the optimality of the exponential functions for some minimax problems. *SIAM Journal of Applied Math.*, 30(2):324–348, 1976.

[Hip94] Ch. Hipke. Online-Algorithmen zur kompetitiven Suche in einfachen Polygonen. Master's thesis, Universität Freiburg, 1994.

[IK95] Christian Icking and Rolf Klein. Searching for the kernel of a polygon: A competitive strategy. In *Proc. 11th Annu. ACM Sympos. Comput. Geom.*, pages 258–266, 1995.

[Kle92] R. Klein. Walking an unknown street with bounded detour. *Comput. Geom. Theory Appl.*, 1:325–351, 1992.

[Kle94] J. M. Kleinberg. On-line search in a simple polygon. In *Proc. of 5th ACM-SIAM Symp. on Discrete Algorithms*, pages 8–15, 1994.

[KMSY94] M.-Y. Kao, Y. Ma, M. Sipser, and Y. Yin. Optimal constructions of hybrid algorithms. In *Proc. 5th ACM-SIAM Sympos. Discrete Algorithms*, pages 372–381, 1994.

[KRT93] M.-Y. Kao, J. H. Reif, and S. R. Tate. Searching in an unknown environment: An optimal randomized algorithm for the cow-path problem. In *Proc. 4th ACM-SIAM Sympos. Discrete Algorithms*, pages 441–447, 1993.

[LO96] A. López-Ortiz. *On-line Searching on Bounded and Unbounded Domains*. PhD thesis, Department of Computer Science, University of Waterloo, 1996.

[LOS95] A. López-Ortiz and S. Schuierer. Going home through an unknown street. In S. G. Akl, F. Dehne, and J.-R. Sack, editors, *Proc. 4th Workshop on Algorithms and Data Structures*, pages 135–146. LNCS 955, 1995.

[LOS96] A. López-Ortiz and S. Schuierer. Generalized streets revisited. In M. Serna J. Diaz, editor, *Proc. 4th European Symposium on Algorithms*, pages 546–558. LNCS 1136, 1996.

[MI94] A. Mei and Y. Igarashi. Efficient strategies for robot navigation in unknown environment. In *Proc. of 21st Intl. Colloquium on Automata, Languages and Programming*, page ??, 1994.

[PY89] C. H. Papadimitriou and M. Yannakakis. Shortest paths without a map. In *Proc. 16th Internat. Colloq. Automata Lang. Program.*, volume 372 of *Lecture Notes in Computer Science*, pages 610–620. Springer-Verlag, 1989.

[Sch97] S. Schuierer. Lower bounds in on-line geometric searching. In *Proc. of the 11th Intern. Symp. on Fundamentals of Computation Theory*, pages 429–440. LNCS 1279, 1997.

[ST85] D. D. Sleator and R. E. Tarjan. Amortized efficiency of list update and paging rules. *Communications of the ACM*, 28:202–208, 1985.

[Yin94] Y. Yin. *Teaching, Learning, and Exploration*. PhD thesis, Department of Mathematics and Laboratory for Computer Science, MIT, 1994.